Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin Spaces and Applications to Problems in Partial Differential Equations

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Dedicated to Michael Cwikel on the occasion of his 60-th birthday

1   Introduction

In their ground-breaking work [**42**], D. Jerison and C. Kenig have studied the well-posedness of the Poisson problem for the Dirichlet Laplacian on Besov and Bessel potential spaces,

$$\Delta u = f \in B_\alpha^{p,p}(\Omega), \quad u \in B_{\alpha+2}^{p,p}(\Omega), \quad \operatorname{Tr} u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

$$\Delta u = f \in L_\alpha^p(\Omega), \quad u \in L_{\alpha+2}^p(\Omega), \quad \operatorname{Tr} u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Let $\mathbb{G}_D$ be the Green operator associated with the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^n$. That is, for $f \in C^\infty(\bar{\Omega})$, the function $u := \mathbb{G}_D f \in W^{1,2}(\Omega)$ is the unique solution (given by Lax-Milgram's lemma) of the variational problem

$$\Delta u = f \quad \text{in} \quad \Omega, \qquad \operatorname{Tr} u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

In this terminology, Jerison and Kenig have identified the largest subset $\mathcal{R}_D(\Omega)$ of $[1, \infty] \times \mathbb{R}$ with the property that the operators

$$\nabla^2 \mathbb{G}_D : B_\alpha^{p,p}(\Omega) \longrightarrow B_\alpha^{p,p}(\Omega), \tag{1.4}$$

$$\nabla^2 \mathbb{G}_D : L_\alpha^p(\Omega) \longrightarrow L_\alpha^p(\Omega), \tag{1.5}$$

are bounded for each $(p, \alpha) \in \mathcal{R}_D(\Omega)$. The methods developed by Jerison and Kenig, though beautiful in their elegance and sharpness, rely in an essential fashion on the maximum principle and, as such, do not readily adapt to other natural boundary conditions, e.g., of *Neumann type*. In fact, the latter issue was singled out as open problem #3.2.21 in Kenig's book [**52**]. Subsequently, this has been solved in [**31**] (and further extended in [**61**]) via a new approach which relies on boundary layer potentials.

When $\partial\Omega \in C^\infty$ the operator $\nabla^2 \mathbb{G}_D$ falls under the scope of the classical theory of singular integral operators of Calderón-Zygmund type. In particular, it maps $L^p(\Omega)$ boundedly into itself for any $1 < p < \infty$ – this is the point of view adopted by Agmon, Douglis and Nirenberg in the late 50's ([**1**]). An extension due to Chang, Dafni, Krantz and Stein in the 90's ([**13**], [**12**]) is that, if $\partial\Omega \in C^\infty$, then the aforementioned result continues to hold for $p \leq 1$ provided $L^p(\Omega)$ is replaced by the local Hardy space $h^p(\Omega)$. In particular, they have shown that

$$\nabla^2 \mathbb{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega), \tag{1.6}$$

is bounded if $\partial\Omega \in C^\infty$ and $\frac{n}{n+1} < p \leq 1$. The version of (1.6) corresponding to $\Omega = \mathbb{R}^n$ has been dealt with much earlier by C. Fefferman and E. Stein [32]. The situation is *radically* different in less smooth domains. For example, by further refining a construction due to B. Dahlberg [19], D. Jerison and C. Kenig have produced in [42] an example of a bounded C^1-domain Ω, along with a function $f \in C^\infty(\bar{\Omega})$, such that $\nabla^2 \mathbb{G} f \notin L^1(\Omega)$.

In the case of Lipschitz domains, Jerison and Kenig have also produced nontrivial counterexamples in [42] which show that the range $\mathcal{R}_D(\Omega)$ introduced earlier in connection with (1.4)-(1.5) is optimal *if one insists that $p \geq 1$ (when all spaces involved are Banach)*. However, the Besov scale $B_\alpha^{p,p}(\Omega)$ naturally continues below $p = 1$, though the corresponding spaces are no longer *locally convex*. The consideration of the *entire* scale $B_\alpha^{p,q}(\Omega)$, $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, is also natural because Besov spaces with $p < 1$ offer a natural framework for certain types of numerical approximation schemes (cf. [23], [24], [25], [26]).

Let us also remark that the Bessel potential spaces fit naturally in the Triebel-Lizorkin scale in the sense that $L_\alpha^p(\Omega) = F_\alpha^{p,2}(\Omega)$ if $1 < p < \infty$, and that $F_\alpha^{p,q}(\Omega)$ is defined for $0 < p, q < \infty$, $\alpha \in \mathbb{R}$. Working with this more general scale is only very natural, but also convenient given that the local Hardy spaces $h^p(\Omega)$ occur precisely when $p \leq 1$, $q = 2$ and $\alpha = 0$.

Further research in this regard has been stimulated by a conjecture made by D.-C. Chang, S. Krantz and E. Stein concerning the mapping properties of Green potentials when the underlying domain is less smooth. In reference to the boundedness of (1.6), on. p. 130 of [14] the authors write: "*For some applications it would be desirable to find minimal smoothness conditions on $\partial\Omega$ in order for our analysis of the Dirichlet and Neumann problems to remain valid. We do not know whether $C^{1+\varepsilon}$ boundary is sufficient in order to obtain h^p estimates for the Dirichlet problem when p is near 1.*" They also go on to note that "*The literature for the Dirichlet and Neumann problems for domains with Lipschitz boundaries (see [51]) teaches us that when the boundary is only Lipschitz then one can expect favorable behavior for a restricted range of p. It would be of interest to explore similar phenomena vis à vis the Hardy spaces introduced here.*" Furthermore, on p. 289 of [13] the authors make the conjecture that estimate (1.6) continues to hold if Ω is of class C^k, where $k > 1/p$.

The Chang-Krantz-Stein conjecture has been recently solved in [56] where the following result has been established.

Theorem 1.1. *For every bounded Lipschitz domain Ω in $\mathbb{R}^n$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the property that the operator (1.6) is well-defined and bounded whenever $1 - \varepsilon < p < 1$. Furthermore, an analogous result is valid for the Green potential associated with the Neumann Laplacian.*

Finally, if $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain, then one can take $\frac{n}{n+1} < p < 1$.

This result is rather surprising, particularly in the light of Dahlberg's counterexample, mentioned above, according to which such a result is false for each $p > 1$ even in the class of C^1 domains.

In this paper we would like to focus on those aspects in the proof of Theorem 1.1 where interpolation methods play a crucial role. In order to be able to bring those into focus, consider the following regularity theorem, due to Jerison and Kenig, which is pivotal to the work in [42], where (1.1)-(1.2) have been studied.

Theorem 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and denote by δ the distance function to the boundary of Ω, i.e., $\delta(x) := \mathrm{dist}(x, \partial\Omega)$. Also, fix a nonnegative integer k.*

If $1 \leq p \leq \infty$ and $s \in (0, 1)$, then for any harmonic function u in Ω,

$$u \in B_{k+s}^{p,p}(\Omega) \Longleftrightarrow \delta^{1-s}|\nabla^{k+1}u| + |\nabla^k u| + |u| \in L^p(\Omega). \tag{1.7}$$

Moreover, if $1 < p < \infty$ and $s \in [0, 1]$, then for any harmonic function u in Ω,

$$u \in L_{k+s}^p(\Omega) \Longleftrightarrow \delta^{1-s}|\nabla^{k+1}u| + |\nabla^k u| + |u| \in L^p(\Omega). \tag{1.8}$$

2

This corresponds to Theorems 4.1-4.2 on p. 181 of [**42**], which Jerison and Kenig have proved using deep properties of harmonic functions in Lipschitz domains.

We would like to consider this result from a more general perspective, and we start by introducing some notation. Let $L = \sum_{|\gamma|=m} a_\gamma \partial^\gamma$ be a homogeneous, constant coefficient, elliptic differential operator of order $m \in 2\,\mathbb{N}$ in $\mathbb{R}^n$. For a fixed, bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, denote by $\operatorname{Ker} L$ the space of functions satisfying $Lu = 0$ in Ω. Then, for $0 < p < \infty$ and $\alpha \in \mathbb{R}$, introduce $\mathbb{H}^p_\alpha(\Omega; L)$ the space of functions $u \in \operatorname{Ker} L$ subject to the condition

$$\|u\|_{\mathbb{H}^p_\alpha(\Omega;L)} := \|\delta^{\langle\alpha\rangle - \alpha} |\nabla^{\langle\alpha\rangle} u|\|_{L^p(\Omega)} + \sum_{j=0}^{\langle\alpha\rangle - 1} \|\nabla^j u\|_{L^p(\Omega)} < +\infty. \tag{1.9}$$

Above, ∇^j stands for vector of all mixed-order partial derivatives of order j and $\langle\alpha\rangle$ is the smallest nonnegative integer greater than or equal to α, i.e.,

$$\langle\alpha\rangle := \begin{cases} \alpha, & \text{if } \alpha \text{ is a nonnegative integer,} \\ [\alpha] + 1, & \text{if } \alpha > 0, \ \alpha \notin \mathbb{N}, \\ 0, & \text{if } \alpha < 0, \end{cases} \tag{1.10}$$

where $[\cdot]$ is the integer-part function. Parenthetically, let us point out that an equivalent quasi-norm on $\mathbb{H}^p_\alpha(\Omega; L)$ is given by

$$\|\delta^{\langle\alpha\rangle - \alpha} |\nabla^{\langle\alpha\rangle} u|\|_{L^p(\Omega)} + \sup_{x \in \mathcal{O}} |u(x)|, \tag{1.11}$$

where $\mathcal{O}$ denotes some fixed compact subset of Ω.

With Ω and L as above, for each $\alpha \in \mathbb{R}$ and each $0 < p < \infty$, consider now the formulas

$$\mathbb{H}^p_\alpha(\Omega; L) = F^{p,2}_\alpha(\Omega) \cap \operatorname{Ker} L, \tag{1.12}$$

$$\mathbb{H}^p_\alpha(\Omega; L) = B^{p,p}_\alpha(\Omega) \cap \operatorname{Ker} L. \tag{1.13}$$

Theorem 1.2 can then be restated by saying that if Ω is a bounded Lipschitz domain and $L = \Delta$ then (1.13) holds if $1 \le p \le \infty$ and $\alpha > 0$, whereas (1.12) holds provided $1 < p < \infty$ and $\alpha \ge 0$.

In [**56**], the following generalization of Theorem 1.2, which has played a crucial role in the solution of the Chang-Krantz-Stein conjecture, has been proved.

Theorem 1.3. *Assume that L is a homogeneous, constant coefficient, elliptic differential operator and that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then (1.12)-(1.13) are valid for every $\alpha \in \mathbb{R}$ and $0 < p < \infty$.*

The strategy for proving Theorem 1.3 consists of two basic steps.

Step I. Show that the equalities (1.12)-(1.13) hold for certain special values of the indices p, α.

Step II. Use interpolation to extend the range of validity of (1.12)-(1.13) to $\alpha \in \mathbb{R}$, $p \in (0, \infty)$.

In this paper, we are going to restrict our attention to Step II which relies heavily on ideas from interpolation theory. The crux of the matter is establishing that the scales of spaces intervening in (1.12)-(1.13) are stable under interpolation. In this regard, we mention the following

Theorem 1.4. *Let L be a homogeneous, constant-coefficient, elliptic differential operator and suppose that Ω is a bounded, star-like Lipschitz domain in $\mathbb{R}^n$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p < \infty$, $0 < \theta < 1$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then the following interpolation formulas hold with equivalent norms:*

$$\left(\mathbb{H}^p_{\alpha_0}(\Omega; L), \, \mathbb{H}^p_{\alpha_1}(\Omega; L) \right)_{\theta,p} = \mathbb{H}^p_\alpha(\Omega; L), \tag{1.14}$$

$$\left[\mathbb{H}^p_{\alpha_0}(\Omega; L), \, \mathbb{H}^p_{\alpha_1}(\Omega; L) \right]_\theta = \mathbb{H}^p_\alpha(\Omega; L). \tag{1.15}$$

Above, $(\cdot,\cdot)_{\theta,p}$ and $[\cdot,\cdot]_\theta$ stand, respectively, for the real and the complex method of interpolation.

Theorem 1.5. *Consider an elliptic, homogeneous, constant coefficient differential operator L and fix a bounded Lipschitz domain Ω in $\mathbb{R}^n$. Also, assume that $0 < q_0, q_1, q \leq \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < \theta < 1$, and set $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then, if $0 < p < \infty$,*

$$\left(F_{\alpha_0}^{p,q_0}(\Omega) \cap \operatorname{Ker} L \,,\, F_{\alpha_1}^{p,q_1}(\Omega) \cap \operatorname{Ker} L\right)_{\theta,q} = B_\alpha^{p,q}(\Omega) \cap \operatorname{Ker} L, \tag{1.16}$$

and if $0 < p \leq \infty$,

$$\left(B_{\alpha_0}^{p,q_0}(\Omega) \cap \operatorname{Ker} L \,,\, B_{\alpha_1}^{p,q_1}(\Omega) \cap \operatorname{Ker} L\right)_{\theta,q} = B_\alpha^{p,q}(\Omega) \cap \operatorname{Ker} L. \tag{1.17}$$

Next, let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then

$$\left[F_{\alpha_0}^{p_0,q_0}(\Omega) \cap \operatorname{Ker} L \,,\, F_{\alpha_1}^{p_1,q_1}(\Omega) \cap \operatorname{Ker} L\right]_\theta = F_\alpha^{p,q}(\Omega) \cap \operatorname{Ker} L. \tag{1.18}$$

Finally, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ and either $p_0 + q_0 < \infty$ or $p_1 + q_1 < \infty$, then

$$\left[B_{\alpha_0}^{p_0,q_0}(\Omega) \cap \operatorname{Ker} L \,,\, B_{\alpha_1}^{p_1,q_1}(\Omega) \cap \operatorname{Ker} L\right]_\theta = B_\alpha^{p,q}(\Omega) \cap \operatorname{Ker} L, \tag{1.19}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Essentially, Theorem 1.5 asserts that being a null-solution of a (fixed) elliptic differential operator commutes with real and complex interpolation on Besov and Triebel-Lizorkin scales. The starting point in the proof is establishing (1.16)-(1.19) in the degenerate case $L = 0$, when $\operatorname{Ker} L$ can be dropped. That the scales $F_\alpha^{p,q}(\Omega)$, $B_\alpha^{p,q}(\Omega)$ are amenable to real and complex interpolation follows from the corresponding results for $\Omega = \mathbb{R}^n$ and a deep result due to V. Rychkov ([**74**]) to the effect that there exists a linear universal extension operator, mapping distributions in the Lipschitz domain Ω to tempered distributions in $\mathbb{R}^n$, with preservation of smoothness (measured on the Triebel-Lizorkin and Besov scales).

How the scales $F_\alpha^{p,q}(\mathbb{R}^n)$ and $B_\alpha^{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, behave under the real method of interpolation is something which has been well-understood for a long time. Excellent references on this matter are [**82**], [**4**]. The attractive feature of the real method of interpolation is that it allows the consideration of quasi-normed spaces, which is the case with the aforementioned scales when $\min\{p,q\} < 1$.

By way of contrast, the complex method of interpolation (at least in its original inception by A.P. Calderón and J.L. Lions), requires that the spaces in question be Banach. This limitation has, over the years, motivated the introduction of a number of ad-hoc remedies, such as those put forward in [**72**], [**82**], [**41**]. A more recent point of view, emerging from work in [**47**] and then tailored more precisely in [**58**] to the specific nature of the Triebel-Lizorkin and Besov scales in $\mathbb{R}^n$, is that for a certain class of quasi-Banach spaces (large enough to contain the Besov and Triebel-Lizorkin scales), the complex method of interpolation continues to work in its original design. In this paper, we devote ample space to this issue and present (perhaps for the first time) a detailed and fairly self-contained account. This has independent interest and our hope is that specialists in other fields (such as harmonic analysis, partial differential equations, etc.), will find it useful. Along the way, we make an effort to link this discussion to other aspects of practical interest (not necessarily directly related to the topics discussed up to this point) where interpolation methods play an important role. Here we only wish to indicate their location, while reviewing the contents of the paper.

In Section 2 we have amassed the most basic definitions and discussed notation and conventions used throughout the paper. Section 3 is dedicated to reviewing some classical scales of smoothness spaces in $\mathbb{R}^n$, such as Bessel potential spaces, Sobolev spaces, Hölder spaces, Hardy spaces, spaces of functions of bounded mean oscillations, Besov spaces, and Triebel-Lizorkin spaces. Various identifications amongst these classes are presented. Section 4 deals with mapping properties for certain of pseudodifferential operators on Besov and Triebel-Lizorkin spaces in $\mathbb{R}^n$. In Section 5 we present a brief review of standard interpolation results for the scales of Besov and Triebel-Lizorkin in $\mathbb{R}^n$. In Section 6 we introduce smoothness spaces, which are analogous in nature to those in §3, on arbitrary Lipschitz domains, and also discuss various identification results. In Section 7 we review the general setup for the complex method of interpolation for quasi-Banach spaces. We highlight the importance of the concept of analytic convexity and prove several useful abstract interpolation results. In Section 8 we discuss a general stability result, and sketch a number of relevant applications to PDE's. We revisit in Section 9 the issue of complex interpolation for Besov and Triebel-Lizorkin spaces for the full range of indices for which these spaces are defined. This is first done in $\mathbb{R}^n$, then in arbitrary Lipschitz domains. Section 10 contains a discussion on how compactness is preserved and extrapolated on scales of Besov and Triebel-Lizorkin spaces. The section ends with an application to boundary value problems in C^1 domains. As a preamble to the proofs of Theorems 1.4-1.5, in Section 11 we collect several useful weighted norm inequalities for solutions of elliptic PDE's in Lipschitz domains. Finally, in Section 12, we present the proofs of Theorems 1.4-1.5.

2   Notation and conventions

As is customary, $\mathbb{Z}$ and $\mathbb{N}$ are, respectively, the collection of integers and the collection of positive integers. We also set $\mathbb{N}_o := \mathbb{N} \cup \{0\}$. Throughout the paper, $\mathbb{R}^n$ stands for the standard Euclidean space, equipped with the canonical orthonormal basis $e_j := (\delta_{jk})_k$, $1 \leq j \leq n$, and norm $|x| := \sqrt{x_1^2 + \cdots x_n^2}$, if $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Given a (measurable) set E in $\mathbb{R}^n$, we denote by $|E|$ its measure and by χ_E its characteristic function. The (open) ball centered at $a \in \mathbb{R}^n$ and having radius $r > 0$ is going to be denoted by $B(a, r)$.

By a cube Q in $\mathbb{R}^n$ we shall always mean a set of the form $I_1 \times I_2 \times \cdots I_n$ where the I_j's are intervals of the same length, denoted $l(Q)$. If Q is a cube and $\lambda > 0$, we let λQ stand for the cube in $\mathbb{R}^n$ concentric with Q and whose side length is $\lambda l(Q)$.

By $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$ we denote the j-th partial derivative in $\mathbb{R}^n$, $1 \leq j \leq n$, and by $\nabla = (\partial_1, ..., \partial_n)$ the gradient operator. Iterated partial derivatives are denoted by $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index with nonnegative integer components, of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For a scalar-valued function f, $\nabla^k f$ is the vector of all partial derivatives of order k of f, i.e., $\nabla^k f = (\partial^\alpha f)_{\alpha: |\alpha| = k}$.

Given $0 < p \leq \infty$, we let p' stand for the Hölder conjugate exponent of p when $1 \leq p \leq \infty$, and set $p' := \infty$ otherwise. For a number $s \in \mathbb{R}$, $[s]$ denotes its integer part, while $(s)_+$ stands for max $\{s, 0\}$.

We let $\langle \cdot, \cdot \rangle$ stand for the duality pairing between a topological vector space X and its dual X^* (which should be clear from the context). In addition, we use the same piece of notation to denote the inner product in various Hilbert spaces, including $\mathbb{R}^n$.

Given an open set $\Omega \subseteq \mathbb{R}^n$, we denote by $C^0(\Omega)$ the space of continuous real valued functions on Ω, by $C^r(\Omega)$, $r \in \mathbb{N}$, the space of r times continuously differentiable real valued functions on Ω, and set $C^\infty(\Omega) = \bigcap C^r(\Omega)$ with intersection taken over all $r \in \mathbb{N}$. Next denote by $C_c^r(\Omega)$ the space of functions $\phi \in C^r(\Omega)$ with compact support, and by $D'(\Omega)$ the space of distributions in Ω, i.e. the dual of $C_c^\infty(\Omega)$

equipped with the inductive limit topology. The support of $u \in D'(\Omega)$, denoted by supp (u), is the set of points in Ω having no open neighborhood in which u vanishes. As is customary, we let $S(\mathbb{R}^n)$ denote the Schwartz class, of smooth, rapidly decreasing functions, and by $S'(\mathbb{R}^n)$ the space of tempered distributions in $\mathbb{R}^n$. All partial derivatives in this paper are considered in the sense of distributions. The Laplacian is then given by $\Delta = \partial_1^2 + \cdots + \partial_n^2$.

By $\mathcal{F}$ we denote the Fourier transform, mapping the space of tempered distributions to itself, and let $\mathcal{F}^{-1}$ stand for its inverse.

Given a subspace $X(\Omega)$ of distributions in the open set $\Omega \subseteq \mathbb{R}^n$ we denote by $X_{\mathrm{loc}}(\Omega)$ the space of distributions u in Ω such that $\xi u \in X(\Omega)$ for every $\xi \in C_c^\infty(\Omega)$. If Ω is an arbitrary open subset of $\mathbb{R}^n$, we denote by $f|_\Omega \in D'(\Omega)$ the restriction of a distribution $f \in D'(\mathbb{R}^n)$ to Ω.

Let $0 < p \le \infty$ and assume that $\mathcal{Q}$ is a countable set. By $\ell^p(\mathcal{Q})$ we denote the space of all numerical sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{Q}}$ such that $\|\lambda\|_{\ell^p(\mathcal{Q})} := \left(\sum_{Q \in \mathcal{Q}} |\lambda_Q|^p \right)^{1/p} < \infty$ if $p < \infty$, with the usual convention when $p = \infty$. When $\mathcal{Q} = \mathbb{N}_o$, we abbreviate $\ell^p := \ell^p(\mathbb{N}_o)$.

Throughout the paper, $A \approx B$ signifies that the quotient A/B is bounded away from zero and infinity, by finite, positive constants which are independent of the relevant parameters in A, B. Finally, we adopt the standard practice of denoting by C generic constants which may differ from one occurrence to the other and write $C = C(\kappa)$ whenever it is important to stress that C depends on a certain parameter κ.

3 Function spaces on $\mathbb{R}^n$

We debut by recalling that, for each $1 < p < \infty$ and $s \in \mathbb{R}$, the Bessel potential space $L_s^p(\mathbb{R}^n)$ is defined by

$$
\begin{aligned}
L_s^p(\mathbb{R}^n) &:= \left\{ (I - \Delta)^{-s/2} g : g \in L^p(\mathbb{R}^n) \right\} \\
&= \left\{ \mathcal{F}^{-1}(1 + |\xi|^2)^{-s/2} \mathcal{F} g : g \in L^p(\mathbb{R}^n) \right\}
\end{aligned}
\tag{3.1}
$$

and is equipped with the norm

$$
\|f\|_{L_s^p(\mathbb{R}^n)} := \|\mathcal{F}^{-1}(1 + |\xi|^2)^{s/2} \mathcal{F} f\|_{L^p(\mathbb{R}^n)}.
\tag{3.2}
$$

As is well-known, when the smoothness index is a natural number, say $s = k \in \mathbb{N}$, this can be identified with the classical Sobolev space

$$
W^{k,p}(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{W^{k,p}(\mathbb{R}^n)} := \sum_{|\gamma| \le k} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)} < +\infty \right\},
\tag{3.3}
$$

i.e.,

$$
L_k^p(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n), \qquad k \in \mathbb{N}_o, \ 1 < p < \infty.
\tag{3.4}
$$

For further reference, we define here the Hölder space $C^s(\mathbb{R}^n)$, $s > 0$, $s \notin \mathbb{N}$, consisting of functions f for which

$$
\|f\|_{C^s(\mathbb{R}^n)} := \sum_{|\alpha| \le [s]} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\alpha| = [s]} \sup_{x \ne y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^{s - [s]}} < +\infty.
\tag{3.5}
$$

Next we turn our attention to Hardy-type spaces in $\mathbb{R}^n$. Fix a function $\psi \in C_c^\infty(\mathbb{R}^n)$ with supp $(\psi) \subset \{x \in \mathbb{R}^n : |x| < 1\}$ and $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$, and set $\psi_t(x) := t^{-n}\psi(x/t)$ for each $t > 0$. Given a tempered

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distribution $u \in S'(\mathbb{R}^n)$ we define its radial maximal function and its truncated version, respectively, by setting

$$u^{++} := \sup_{0 < t < \infty} |\psi_t * u|, \qquad u^+ := \sup_{0 < t < 1} |\psi_t * u|. \tag{3.6}$$

For $0 < p < \infty$, the classical *homogeneous* Hardy space $H^p(\mathbb{R}^n)$, and its *local* version, $h^p(\mathbb{R}^n)$, introduced in [**39**], are then defined as

$$H^p(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) : \|u\|_{H^p(\mathbb{R}^n)} := \|u^{++}\|_{L^p(\mathbb{R}^n)} < \infty\}, \tag{3.7}$$

$$h^p(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) : \|u\|_{h^p(\mathbb{R}^n)} := \|u^+\|_{L^p(\mathbb{R}^n)} < \infty\}. \tag{3.8}$$

Different choices of the function ψ yield equivalent quasi-norms so (3.7), (3.8) viewed as topological spaces, are intrinsically defined. Also,

$$H^p(\mathbb{R}^n) = h^p(\mathbb{R}^n) = L^p(\mathbb{R}^n), \qquad 1 < p < \infty. \tag{3.9}$$

In analogy with (3.3), Hardy-based Sobolev spaces $h_k^p(\mathbb{R}^n)$, $0 < p < \infty$, $k \in \mathbb{N}_o$, are then defined as

$$h_k^p(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) : \partial^\gamma u \in h^p(\mathbb{R}^n), \ \forall \gamma \in \mathbb{N}_o^n \text{ with } |\gamma| \le k\}, \tag{3.10}$$

and are equipped with the quasi-norm $\|u\|_{h_k^p(\mathbb{R}^n)} := \sum_{|\gamma| \le k} \|\partial^\gamma u\|_{h^p(\mathbb{R}^n)}$. For $0 < p < \infty$ and $k \in \mathbb{N}$ we also set

$$h_{-k}^p(\mathbb{R}^n) := \left\{ u \in S'(\mathbb{R}^n) : u = \sum_{|\gamma| \le k} \partial^\gamma u_\gamma, \ u_\gamma \in h^p(\mathbb{R}^n) \ \forall \gamma \in \mathbb{N}_o^n \text{ with } |\gamma| \le k \right\} \tag{3.11}$$

which we equip with the natural quasi-norm $\|u\|_{h_{-k}^p(\mathbb{R}^n)} := \inf \sum_{|\gamma| \le k} \|u_\gamma\|_{h^p(\mathbb{R}^n)}$, where the infimum is taken over all representations of u.

A function $f \in L_{\text{loc}}^2(\mathbb{R}^n)$ belongs to the space $\text{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} < \infty, \tag{3.12}$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $f_Q := \frac{1}{|Q|} \int_Q f(x) \, dx$. As is well-known, $(H^1(\mathbb{R}^n))^* = \text{BMO}(\mathbb{R}^n)$ (see [**32**]). The local version of $\text{BMO}(\mathbb{R}^n)$ is defined as follows. A function $f \in L_{\text{loc}}^2(\mathbb{R}^n)$ belongs to $\text{bmo}(\mathbb{R}^n)$ if the quantity

$$\|f\|_{\text{bmo}(\mathbb{R}^n)} := \sup \left\{ \sup_{Q : l(Q) \le 1} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2}, \ \sup_{Q : l(Q) > 1} \left(\frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{1/2} \right\} \tag{3.13}$$

is finite. Then $(h^1(\mathbb{R}^n))^* = \text{bmo}(\mathbb{R}^n)$ (see [**39**]).

We now briefly review Besov and Triebel-Lizorkin scales in $\mathbb{R}^n$. The classical Littlewood-Paley definition of Triebel-Lizorkin and Besov spaces (see, for example, [**73**], [**83**]) has the following form. Let Ξ be the collection of all systems $\{\zeta_j\}_{j=0}^\infty \subset S$ with the properties

(i) there exist positive constants A, B, C such that

$$\begin{cases} \text{supp}\,(\zeta_0) \subset \{x : |x| \le A\}, \\ \text{supp}\,(\zeta_j) \subset \{x : B2^{j-1} \le |x| \le C2^{j+1}\} & \text{if } j = 1, 2, 3..., \end{cases} \tag{3.14}$$

(ii) for every multi-index α there exists a positive number c_α such that

$$\sup_{x \in \mathbb{R}^n} \sup_{j \in \mathbb{N}} 2^{j|\alpha|} |\partial^\alpha \zeta_j(x)| \le c_\alpha, \tag{3.15}$$

(iii) there holds

$$\sum_{j=0}^{\infty} \zeta_j(x) = 1 \quad \text{for every} \ \ x \in \mathbb{R}^n. \tag{3.16}$$

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$ and fix some family $\{\zeta_j\}_{j=0}^{\infty} \in \Xi$. If $0 < p < \infty$ then the Triebel-Lizorkin spaces are defined as

$$F_s^{p,q}(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n): \quad \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{j=0}^{\infty} |2^{sj} \mathcal{F}^{-1}(\zeta_j \mathcal{F} f)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\}. \tag{3.17}$$

See [**36**] for a precise definition of $F_s^{\infty,q}(\mathbb{R}^n)$ (cf. also [**73**]). If $0 < p \leq \infty$ then the Besov spaces are defined as

$$B_s^{p,q}(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n): \quad \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} \|2^{sj} \mathcal{F}^{-1}(\zeta_j \mathcal{F} f)\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty \right\}. \tag{3.18}$$

A different choice of the system $\{\zeta_j\}_{j=0}^{\infty} \in \Xi$ yields the same spaces (3.17)-(3.18), albeit equipped with equivalent norms.

There is an alternative version of definitions above starting with a function $\phi_0 \in S(\mathbb{R}^n)$ such that

$$\begin{cases} \int_{\mathbb{R}^n} \phi_0(x) \, dx \neq 0, \\ \phi(x) := \phi_0(x) - 2^{-n} \phi_0(x/2) \Longrightarrow L(\phi) \geq [s], \end{cases} \tag{3.19}$$

where $s \in \mathbb{R}$ and $L(\phi)$ stands for the order up to which the moments of the function ϕ vanish, i.e.

$$\int_{\mathbb{R}^n} x^\alpha \phi(x) \, dx = 0 \quad \text{if} \quad |\alpha| \leq L(\phi). \tag{3.20}$$

It is well-known that, given any $s \in \mathbb{R}$, there exist functions ϕ_0 satisfying (3.19). Indeed, for $s < 1$ the second condition in (3.19) is trivial, whereas for $s \geq 1$ we may take any function $\phi_0 \in S(\mathbb{R}^n)$ whose Fourier transform is $1 + O(|\xi|^{[s]+1})$ uniformly as $\xi \to 0$. Next, let $\phi_j(x) := 2^{jn} \phi(2^j x)$, $j \in \mathbb{N}$, denote the dyadic dilates of ϕ. The Triebel-Lizorkin space $F_s^{p,q}(\mathbb{R}^n)$ is then defined for $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, as

$$F_s^{p,q}(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n): \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{j=0}^{\infty} |2^{sj} \phi_j * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\}. \tag{3.21}$$

In a similar spirit, Besov spaces are defined for $0 < p, q \leq \infty$, $s \in \mathbb{R}$, by

$$B_s^{p,q}(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n): \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} \|2^{sj} \phi_j * f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty \right\}. \tag{3.22}$$

Once again, a different choice of the function $\phi_0 \in S(\mathbb{R}^n)$ satisfying (3.19) yields the same spaces (3.21)-(3.22) with equivalent norms (see, e.g., the discussion in [**74**]). It has to be noted that the two definitions are equivalent, for one can take a function ϕ_0 such that $\zeta_0(\xi) = \mathcal{F}\phi_0(\xi)$ is identically 1 if $|\xi| \leq 1$ and vanishes if $|\xi| \geq 2$. Then with the notation as above ϕ and $\zeta_j = \mathcal{F}\phi_j$ satisfy the required properties.

As is well-known (see, e.g., §2.3.3 in [82]), $B_s^{p,q}(\mathbb{R}^n)$ is a quasi-Banach space for $s \in \mathbb{R}$, $0 < p, q \leq \infty$ (Banach space if $1 \leq p, q \leq \infty$) and

$$S(\mathbb{R}^n) \hookrightarrow B_s^{p,q}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n). \tag{3.23}$$

in the sense of continuous (topological) embeddings. Similarly, $F_s^{p,q}(\mathbb{R}^n)$ is a quasi-Banach space for $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ (Banach space if $1 \leq p < \infty$, $1 \leq q \leq \infty$) and

$$S(\mathbb{R}^n) \hookrightarrow F_s^{p,q}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n). \tag{3.24}$$

Furthermore, $S(\mathbb{R}^n)$ is dense in $B_s^{p,q}(\mathbb{R}^n)$ and $F_s^{p,q}(\mathbb{R}^n)$ provided $s \in \mathbb{R}$ and $0 < p, q < \infty$. The class of Schwartz functions is *not* dense in $B_s^{\infty,q}(\mathbb{R}^n)$, $B_s^{p,\infty}(\mathbb{R}^n)$ and $F_s^{p,\infty}(\mathbb{R}^n)$ if $0 < q \leq \infty$ and $0 < p < \infty$.

Let $\mathcal{Q}_n$ stand for the standard family of dyadic cubes in $\mathbb{R}^n$, i.e.,

$$\mathcal{Q}_n := \{Q_{jk} = 2^{-j}[0,1]^n + k : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \tag{3.25}$$

and define

$$\mathcal{Q}_n^* := \{Q \in \mathcal{Q}_n : \ell(Q) \leq 1\}, \tag{3.26}$$

Also, for an arbitrary function ψ in $\mathbb{R}^n$ and $Q \in \mathcal{Q}_n$ set

$$\psi_Q(x) := 2^{jn}\psi(2^j x - k) \quad \text{if} \quad Q = Q_{jk}, \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^n. \tag{3.27}$$

Following [36], [38] the spaces $b_s^{p,q}$ and $f_s^{p,q}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$, consist of sequences $\{\lambda_Q\}_{Q \in \mathcal{Q}_n^*}$ equipped with the quasi-norms

$$\|\{\lambda_Q\}_{Q \in \mathcal{Q}_n^*}\|_{b_s^{p,q}} := \left(\sum_{j=0}^{\infty} \Big(\sum_{\substack{Q \in \mathcal{Q}_n^* \\ l(Q) = 2^{-j}}} [\,|Q|^{-s/n-1/2+1/p}|\lambda_Q|\,]^p \Big)^{q/p} \right)^{1/q}, \tag{3.28}$$

$$\|\{\lambda_Q\}_{Q \in \mathcal{Q}_n^*}\|_{f_s^{p,q}} := \left\| \Big(\sum_{\substack{Q \in \mathcal{Q}_n \\ l(Q) \leq 1}} (|Q|^{-s/n-1/2}|\lambda_Q|\chi_Q)^q \Big)^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \tag{3.29}$$

provided $p < \infty$. There is also an appropriate version of this definition when $p = \infty$. Specifically,

$$\|\{\lambda_Q\}_{Q \in \mathcal{Q}_n^*}\|_{f_s^{\infty,q}} := \sup_{P \in \mathcal{Q}_n^*} \left(\frac{1}{|P|} \int_P \sum_{Q \in \mathcal{Q}_n, Q \subset P} \Big(|Q|^{-1/2-s/n}|\lambda_Q|\chi_Q(x)\Big)^q dx \right)^{1/q}$$

$$= \sup_{P \in \mathcal{Q}_n^*} \left(\frac{1}{|P|} \sum_{Q \in \mathcal{Q}_n, Q \subset P} \Big(|Q|^{-1/2-s/n}|\lambda_Q|\Big)^q |Q| \right)^{1/q}. \tag{3.30}$$

Observe that $J_\alpha(\{\lambda_Q\}_{Q \in \mathcal{Q}_n}) := \{|Q|^{-\alpha/n}\lambda_Q\}_{Q \in \mathcal{Q}_n}$ is an isomorphism between $f_s^{p,q}$ and $f_{s+\alpha}^{p,q}$ for each $\alpha \in \mathbb{R}$. Also, as is well known, each $f_s^{p,q}$ is a quasi-Banach lattice.

We next review the wavelet characterization of the Besov and Triebel-Lizorkin spaces. For every choice of $r \in \mathbb{N}$ and $L \in \mathbb{N} \cup \{0, -1\}$ it is possible to construct

a "father" wavelet $\varphi \in C_c^r(\mathbb{R}^n)$ and a family of
"mother" wavelets $\psi^\ell \in C_c^r(\mathbb{R}^n)$, $\ell = 1, 2, ..., 2^n - 1$, $\tag{3.31}$

in the sense of [21], [55], [59], such that $L(\psi^\ell) \geq L$ for each $1 \leq \ell \leq 2^n - 1$, i.e.

$$\int_{\mathbb{R}^n} x^\alpha \psi^\ell(x)\, dx = 0 \quad \text{whenever} \quad |\alpha| \leq L, \ 1 \leq \ell \leq 2^n - 1. \tag{3.32}$$

In the sequel, we shall refer to these as *Daubechies wavelets*. Another variant of this construction, corresponding to the so-called *Lemarie-Meyer wavelets*, allows for $\varphi, \psi^\ell \in S(\mathbb{R}^n)$ and $L = \infty$. Cf. [59] for details. The most striking property of the wavelet functions is that

$$\{\varphi_Q : Q \in \mathcal{Q}_n, \ l(Q) = 1\} \cup \{\psi_Q^\ell : Q \in \mathcal{Q}_n, \ l(Q) \leq 1, \ 1 \leq \ell \leq 2^n - 1\}, \tag{3.33}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$. For uniformity of notation set $\psi_Q^{2^n} := \varphi_Q$. The *wavelet coefficients* of a given tempered distribution $f \in S'(\mathbb{R}^n)$ relative to a family of Daubechies or Lemarie-Meyer wavelets, $\varphi, \psi^\ell, \ \ell = 1, ..., 2^n - 1$, are then defined as

$$\lambda_Q^\ell(f) := \langle f, \psi_Q^\ell \rangle, \quad \text{for } Q \in \mathcal{Q}_n^* \text{ and } \ell = 1, ..., 2^n. \tag{3.34}$$

The wavelet characterization of Besov and Triebel-Lizorkin spaces we are about to describe next goes back to [34], [35], [38].

Recall that the collection of vectors $x_0, x_1, ...,$ in the space $(X, \|\cdot\|_X)$ is called Schauder basis if every vector $x \in X$ can be written in the form

$$x = \sum_{j=0}^\infty \lambda_j x_j, \tag{3.35}$$

where $\lambda_i \in \mathbb{R}$, $j = 0, 1, ...,$ the series (3.35) converges in the $\|\cdot\|_X$ norm, i.e.

$$\lim_{N \to \infty} \left\| x - \sum_{j=0}^N \lambda_j x_j \right\|_X = 0, \tag{3.36}$$

and coefficients $\lambda_j, \ j = 0, 1, ...,$ are uniquely determined by (3.35)-(3.36). The Schauder basis is unconditional if for every $x \in X$ the series (3.35) converges unconditionally to x, that is there exists $\varepsilon > 0$ and F_ε, a finite subset of $\mathbb{N}$, such that $\|x - \sum_{j \in F} \lambda_j x_j\|_X \leq \varepsilon$ for every finite set $F \in \mathbb{N}_o$ containing F_ε.

Theorem 3.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and assume that $r \geq ([s] + 1)_+$ and $L \geq \max\{[J - n - s], -1\}$, where $J := \frac{n}{\min\{1, p, q\}}$. Then for any family of Daubechies wavelets as in (3.31) the following is true. For each $f \in S'(\mathbb{R}^n)$ one has*

$$f \in F_s^{p,q}(\mathbb{R}^n) \iff \{\lambda_Q^\ell(f)\}_{Q \in \mathcal{Q}_n^*} \in f_s^{p,q} \text{ for each } \ell = 1, ..., 2^n, \tag{3.37}$$

with a naturally accompanying norm estimate.

Next, assume that $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $r \geq ([s] + 1)_+$ and $L \geq \max\{[J - n - s], -1\}$, where $J := \frac{n}{\min\{1, p\}}$. Then for any family of Daubechies wavelets as in (3.31) the following is true. For each $f \in S'(\mathbb{R}^n)$ one has

$$f \in B_s^{p,q}(\mathbb{R}^n) \iff \{\lambda_Q^\ell(f)\}_{Q \in \mathcal{Q}_n^*} \in b_s^{p,q} \text{ for each } \ell = 1, ..., 2^n, \tag{3.38}$$

with a naturally accompanying norm estimate.

Finally, similar statements are true in the case when the Daubechies wavelets are replaced by Lemarie-Meyer wavelets, this time, with no restrictions on r (regularity) and L (number of vanishing moments). In either case, (3.33) is an unconditional Schauder basis in $F_s^{p,q}(\mathbb{R}^n)$ if $q < \infty$, and in $B_s^{p,q}(\mathbb{R}^n)$ if $\max\{p, q\} < \infty$.

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More detailed accounts can be found in, e.g., [**34**], [**35**], [**37**], [**38**], [**82**], [**83**], [**73**], [**70**]. There are also homogeneous versions of the Besov and Triebel-Lizorkin scales, denoted by $\dot{B}_p^{s,q}(\mathbb{R}^n)$ and $\dot{F}_p^{s,q}(\mathbb{R}^n)$, respectively. To define them, fix a Schwartz function φ such that:

1. $\operatorname{supp}\mathcal{F}(\varphi) \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and $|\mathcal{F}(\varphi)(\xi)| \geq c > 0$ uniformly for $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$,

2. $\sum_{i \in \mathbb{Z}} |\mathcal{F}(\varphi)(2^i \xi)|^2 = 1$ if $\xi \neq 0$.

Then for $s \in \mathbb{R}$ and $0 < p, q \leq +\infty$ the homogeneous Besov spaces are defined as follows:

$$\dot{B}_s^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} := \left(\sum_{i \in \mathbb{Z}} (2^{is} \|\varphi_i * f\|_{L^p})^q \right)^{\frac{1}{q}} < \infty \right\}, \qquad (3.39)$$

whereas, for $s \in \mathbb{R}$, $0 < p < +\infty$ and $0 < q \leq +\infty$, the homogeneous Triebel-Lizorkin spaces are defined as

$$\dot{F}_s^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{i \in \mathbb{Z}} (2^{is} |\varphi_i * f|)^q \right)^{\frac{1}{q}} \right\|_{L^p} < \infty \right\}. \qquad (3.40)$$

There is also an appropriate version of (3.40) corresponding to $p = \infty$; see §5 in [**36**] for a precise definition.

As with their inhomogeneous counterparts, (3.39)–(3.40) are Banach spaces for $p, q \geq 1$, but only quasi-Banach when either $p < 1$ or $q < 1$. Following [**35**], we also introduce a discrete version of the Triebel-Lizorkin scale of spaces by defining $\dot{f}_s^{p,q}$, for $s \in \mathbb{R}$, $0 < p < +\infty$ and $0 < q \leq +\infty$ as the collection of all numerical sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{Q}_n}$ such that

$$\|\lambda\|_{\dot{f}_s^{p,q}} := \left\| \left(\sum_{Q \in \mathcal{Q}_n} (|Q|^{-1/2 - s/n} |\lambda_Q| \chi_Q)^q \right)^{1/q} \right\|_{L^p} < +\infty. \qquad (3.41)$$

There is also an appropriate version of this definition when $p = \infty$. Specifically,

$$\|\lambda\|_{\dot{f}_s^{\infty,q}} := \sup_{P \in \mathcal{Q}_n} \left(\frac{1}{|P|} \int_P \sum_{Q \subset P} (|Q|^{-1/2 - s/n} |\lambda_Q| \chi_Q(x))^q \, dx \right)^{1/q}. \qquad (3.42)$$

Observe that $J_\alpha(\{\lambda_Q\}_Q) := \{|Q|^{-\alpha/n} \lambda_Q\}_Q$ is an isomorphism between $\dot{f}_s^{p,q}$ and $\dot{f}_{s+\alpha}^{p,q}$ for each $\alpha \in \mathbb{R}$. Also, as is well known, each $\dot{f}_s^{p,q}$ is a quasi-Banach lattice. These enjoy similar properties as their inhomogeneous counterparts.

Next, as in §5 of [**38**], the sequence spaces $\dot{b}_s^{p,q}$ associated with the Besov scale are introduced, for $s \in \mathbb{R}$, and $0 < p, q \leq \infty$, as the collection of all numerical sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{Q}_n}$ satisfying

$$\|\lambda\|_{\dot{b}_s^{p,q}} := \left\{ \sum_{j \in \mathbb{Z}} \left(\left\| \sum_{l(Q) = 2^{-j}} |Q|^{-1/2 - s/n} |\lambda_Q| \chi_Q \right\|_{L^p} \right)^q \right\}^{1/q}$$

$$= \left\{ \sum_{j \in \mathbb{Z}} \left(\sum_{l(Q) = 2^{-j}} (|Q|^{-1/2 - s/n + 1/p} |\lambda_Q|)^p \right)^{q/p} \right\}^{1/q} < +\infty. \qquad (3.43)$$

They are all quasi-Banach spaces for the indicated ranges of indices.

It has long been known that many classical smoothness spaces are encompassed by the Besov and Triebel-Lizorkin scales. For example,

$$C^s(\mathbb{R}^n) = B_s^{\infty,\infty}(\mathbb{R}^n), \quad 0 < s \notin \mathbb{Z}, \tag{3.44}$$

$$L^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \tag{3.45}$$

$$L_s^p(\mathbb{R}^n) = F_s^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \qquad s \in \mathbb{R}, \tag{3.46}$$

$$W^{k,p}(\mathbb{R}^n) = F_k^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \qquad k \in \mathbb{N}, \tag{3.47}$$

$$h^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), \quad 0 < p \le 1, \tag{3.48}$$

$$\mathrm{bmo}(\mathbb{R}^n) = F_0^{\infty,2}(\mathbb{R}^n), \tag{3.49}$$

$$\mathrm{BMO}(\mathbb{R}^n) = \dot{F}_0^{\infty,2}(\mathbb{R}^n). \tag{3.50}$$

Furthermore, it has been established in [**56**] that

$$h_k^p(\mathbb{R}^n) = F_k^{p,2}(\mathbb{R}^n), \quad 0 < p \le 1, \ k \in \mathbb{Z}. \tag{3.51}$$

We conclude our review by including a useful lifting result. Assume that $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Then for any $\mu \in \mathbb{R}$,

$$F_s^{p,q}(\mathbb{R}^n) = (I - \Delta)^{\mu/2} F_{s+\mu}^{p,q}(\mathbb{R}^n). \tag{3.52}$$

Also, for any $m \in \mathbb{N}$,

$$
\begin{aligned}
F_s^{p,q}(\mathbb{R}^n) &= \{f \in S'(\mathbb{R}^n) : \partial^\alpha f \in F_{s-m}^{p,q}(\mathbb{R}^n), \ \forall \alpha \text{ with } |\alpha| \le m\} \\
&= \{f \in F_{s-m}^{p,q}(\mathbb{R}^n) : \partial^\alpha f \in F_{s-m}^{p,q}(\mathbb{R}^n), \ \forall \alpha \text{ with } |\alpha| = m\}
\end{aligned}
\tag{3.53}
$$

and

$$
\begin{aligned}
\|f\|_{F_s^{p,q}(\mathbb{R}^n)} &\approx \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{F_{s-m}^{p,q}(\mathbb{R}^n)} \\
&\approx \|f\|_{F_{s-m}^{p,q}(\mathbb{R}^n)} + \sum_{|\alpha| = m} \|\partial^\alpha f\|_{F_{s-m}^{p,q}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.54}
$$

In particular,

$$\partial^\alpha : F_s^{p,q}(\mathbb{R}^n) \longrightarrow F_{s-|\alpha|}^{p,q}(\mathbb{R}^n) \tag{3.55}$$

is bounded. Similar results are valid for the scale of Besov spaces.

4   Mapping properties of pseudodifferential operators

For $0 \le \delta, \rho \le 1$, $m \in \mathbb{R}$, let $S_{\rho,\delta}^m$ be the class of symbols consisting of all functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for each pair of multi-indices β, γ there exists a constant $C_{\beta,\gamma}$ such that

$$|D_\xi^\beta D_x^\gamma p(x,\xi)| \le C_{\beta\gamma}(1 + |\xi|)^{m - \rho|\beta| + \delta|\gamma|}, \tag{4.1}$$

uniformly for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. For $p \in S_{\rho,\delta}^m$ we define the pseudodifferential operator $p(x, D)$ by

$$p(x,D)f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi) \mathcal{F}f(\xi) \, d\xi, \qquad f \in S(\mathbb{R}^n), \tag{4.2}$$

and write $p(x, D) \in OPS_{\rho,\delta}^m$. The following is a consequence of Theorem 6.2.2 on p. 258 of [**83**] (cf. also Remark. 3 on p. 257 of this monograph).

Theorem 4.1. *Let $m \in \mathbb{R}$, $0 \leq \delta < 1$ and fix $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, arbitrary. Then any $T \in OPS_{1,\delta}^m$ induces a bounded, linear operator*

$$T : F_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow F_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{4.3}$$

whenever $0 < p < \infty$. Moreover,

$$T : B_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow B_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{4.4}$$

boundedly, whenever $0 < p \leq \infty$.

We now record a consequence of the above result, particularly useful for treating operators akin to the harmonic Newtonian potential operator.

Corollary 4.2. *Let $a \in S'(\mathbb{R}^n)$ be a tempered distribution for which there exists $R > 0$ such that a is smooth for $|\xi| > R$ and satisfies*

$$|(\partial^\gamma a)(\xi)| \leq C_\gamma |\xi|^{m-|\gamma|}, \qquad |\xi| > R, \quad \gamma \in \mathbb{N}_o^n. \tag{4.5}$$

Assume that $m \in \mathbb{R}$, $m > -n$, and set $T := a(D)$, i.e.

$$Tf(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} a(\xi) \mathcal{F}f(\xi)\, d\xi, \qquad f \in S(\mathbb{R}^n). \tag{4.6}$$

Finally, fix $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ and $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ (viewed below as multiplication operators). Then

$$\phi T \psi : F_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow F_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{4.7}$$

is a bounded operator whenever $0 < p < \infty$. In fact, so is

$$\phi T \psi : B_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow B_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{4.8}$$

if $0 < p \leq \infty$.

Proof. Assume first that (4.5) holds for $\xi \in \mathbb{R}^n \setminus \{0\}$. Fix $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ and note that $u := T(\psi f)$ is a tempered distribution whenever $f \in F_\alpha^{p,q}(\mathbb{R}^n)$. Indeed, $\langle u, g \rangle = (-1)^\mu \langle f, \psi T g \rangle$ and $\psi T g \in C_c^\infty(\mathbb{R}^n)$ for each $g \in \mathcal{S}(\mathbb{R}^n)$. Next, consider some $\theta \in S(\mathbb{R}^n)$ which is identically equal to 1 in a neighborhood of the origin and set $\eta := \mathcal{F}^{-1}\theta \in \mathcal{S}(\mathbb{R}^n)$. Then one can write

$$\mathcal{F}(u) = \theta(\xi) a(\xi) \mathcal{F}(\psi f) + (1 - \theta(\xi)) a(\xi) \mathcal{F}(\psi f) \tag{4.9}$$

and, hence,

$$\phi u = \phi(u * \eta) + \phi \mathcal{F}^{-1}\left((1 - \theta(\xi)) a(\xi) \mathcal{F}(\psi f)\right). \tag{4.10}$$

Since $\mathcal{S}'(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^n)$, the first term in the right-hand side of the above identity belongs to $C_c^\infty(\mathbb{R}^n)$. As for the second one, the membership to $F_{\alpha-m}^{p,q}(\mathbb{R}^n)$ follows from Theorem 4.1 as soon as we observe that $p(x, \xi) := (1 - \theta(\xi)) a(\xi) \in S_{1,0}^m$. This justifies the claim made about (4.7) in the case when (4.5) holds for every $\xi \in \mathbb{R}^n$.

In the case when (4.5) holds as stated, we write $a = a_0 + a_1$ where a_1 is a symbol in $S_{1,0}^m$ and a_0 is a compactly supported distribution. Then $a_1(D)$ has the properties (4.3) and (4.4) by Theorem 4.1 and $\phi a_0(D)\psi$ maps $\mathcal{S}'(\mathbb{R}^n)$ to $C_c^\infty(\mathbb{R}^n)$. The desired conclusion follows. Finally, (4.8) is proved in a similar manner. □

Assume that

$$L = \sum_{|\gamma| = m} a_\gamma \partial^\gamma \tag{4.11}$$

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is a (homogeneous) constant-coefficient, differential operator of order $m \in 2\,\mathbb{N}$ in $\mathbb{R}^n$, which is elliptic, in the sense that there exists a finite constant $\Lambda > 1$ such that if $\sigma(L;\xi) := (-1)^{m/2} \sum_{|\gamma|=m} a_\gamma \xi^\gamma$, then

$$\Lambda^{-1}|\xi|^m \le (-1)^{m/2}\sigma(L;\xi) \le \Lambda|\xi|^m, \qquad \forall\,\xi \in \mathbb{R}^n. \tag{4.12}$$

We have

Theorem 4.3. *If L is a homogeneous, constant coefficient, elliptic operator of order m and $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$, then $\phi L^{-1}\psi$ has mapping properties similar to (4.3)-(4.4).*

Proof. To set the stage, we note that, due to its homogeneity, the symbol $\sigma(L;\xi)^{-1}$ of L^{-1} has a meromorphic extension (as a $S'(\mathbb{R}^n)$-valued mapping) from $\Re e\, m < n$ to $\mathbb{C}$, with poles at $m = n, n+1, \dots,$. A proof can be carried out using the outline given in Exercise 4 on p. 245 in §8 of [**80**], according to which a meromorphic extension can be produced using the transformation $C^\infty(S^{n-1}) \ni w \mapsto W_m \in C^\infty(S^{n-1})$ where, with Γ denoting the Gamma function,

$$W_m(\xi) := (2\pi)^{-n/2} e^{\pi i(m+n)/2} \Gamma(m+n) I_n(m,\xi),$$

$$I_n(m,\xi) := \int\limits_{S^{n-1}} (\omega \cdot \xi + i0)^{-(m+n)} w(\omega)\, dS(\omega). \tag{4.13}$$

Note that $I_n(m,\xi)$ is holomorphic in m for all $m \in \mathbb{C}$. The integrand in $I_n(m,\xi)$ is in $L^1(S^{n-1})$ for $\Re e\, m < -n+1$ and otherwise is a well-defined distribution on S^{n-1} for all m. The poles come from the factor $\Gamma(m+n)$. In the case when w is identically 1, half the poles are canceled due to the zeros of $I_n(m,\xi)$, which in such a case is independent of $\xi \in S^{n-1}$. Finally, for the values $m = n, n+1, \dots,$ one can define the finite part of $\sigma(L;\xi)^{-1}$ as a tempered distribution (see the discussion at the bottom of p. 243 in [**80**], as well as [**49**] for related matters). Having clarified these issue, the desired conclusion now follows from Corollary 4.2. $\qquad\square$

5 Interpolation of Besov and Triebel-Lizorkin spaces: review of known results

Throughout the paper, we let $(\cdot,\cdot)_{\theta,q}$ stand for the standard *real interpolation* bracket. More specifically, consider a compatible couple of quasi-Banach spaces X_0, X_1. Given $a \in X_0 + X_1$ and $0 < t < \infty$ Peetre's K-functional is defined by

$$K(t,a;X_0,X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 \in X_0,\ x_1 \in X_1 \text{ such that } a = x_0 + x_1\}. \tag{5.1}$$

Then we introduce the real interpolation spaces as

$$(X_0,X_1)_{\theta,q} := \left\{ a \in X_0 + X_1 : \|a\|_{(X_0,X_1)_{\theta,q}} := \left(\int_0^\infty (t^{-\theta} K(t,a;X_0,X_1))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}, \tag{5.2}$$

if $0 < \theta < 1$, $0 < q < \infty$, and

$$(X_0,X_1)_{\theta,\infty} := \left\{ a \in X_0 + X_1 : \|a\|_{(X_0,X_1)_{\theta,\infty}} := \sup_{0<t<\infty} t^{-\theta} K(t,a;X_0,X_1) < \infty \right\}, \tag{5.3}$$

for $0 < \theta < 1$.

Various properties of the resulting spaces and more details regarding the real method of interpolation can be found [**4**], [**81**], [**82**]. As far as the real method of interpolation is concerned, we note the following classical result (cf., e.g., Theorem 6.4.5 in [**4**] and [**82**]).

Theorem 5.1. *Let* $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < q_0, q_1, q \leq \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. *Then*

$$(F_{\alpha_0}^{p,q_0}(\mathbb{R}^n), F_{\alpha_1}^{p,q_1}(\mathbb{R}^n))_{\theta,q} = B_\alpha^{p,q}(\mathbb{R}^n), \quad 0 < p < \infty, \tag{5.4}$$

$$(B_{\alpha_0}^{p,q_0}(\mathbb{R}^n), B_{\alpha_1}^{p,q_1}(\mathbb{R}^n))_{\theta,q} = B_\alpha^{p,q}(\mathbb{R}^n), \quad 0 < p \leq \infty. \tag{5.5}$$

Furthermore, similar formulas hold for the homogeneous versions of the Besov and Triebel-Lizorkin spaces.

The complex method of interpolation, denoted by $[\cdot, \cdot]_\theta$, is going to be reviewed in some detail in §7 in a more general setting than that of Banach spaces as it has been originally introduced in [**9**]. At this stage, we would nonetheless like to record the counterpart of Theorem 5.1 for this method, at least for the portion of the Besov and Triebel-Lizorkin scales consisting of Banach spaces. More specifically, we have:

Theorem 5.2. *Let* $\alpha_0, \alpha_1 \in \mathbb{R}$, $1 < p_0, p_1 < \infty$ *and* $1 < q_0, q_1 \leq \infty$. *Then*

$$[F_{\alpha_0}^{p_0,q_0}(\mathbb{R}^n), F_{\alpha_1}^{p_1,q_1}(\mathbb{R}^n)]_\theta = F_\alpha^{p,q}(\mathbb{R}^n), \tag{5.6}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $1 < p_0, p_1, q_0, q_1 \leq \infty$ *and either* $p_0 + q_0 < \infty$ *or* $p_1 + q_1 < \infty$ *then also*

$$[B_{\alpha_0}^{p_0,q_0}(\mathbb{R}^n), B_{\alpha_1}^{p_1,q_1}(\mathbb{R}^n)]_\theta = B_\alpha^{p,q}(\mathbb{R}^n), \tag{5.7}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. *Moreover, analogous formulas are valid for the homogeneous versions of the Besov and Triebel-Lizorkin spaces.*

This particular result is, of course, well-known. See, e.g., [**4**], [**35**], [**73**], [**81**].

6 Function spaces in Lipschitz domains

As a preamble, here we review some basic concepts. Recall that a function $\varphi : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$ is called *Lipschitz* if there exists a finite constant $C > 0$ such that $|\varphi(x) - \varphi(y)| \leq C|x - y|$, for every $x, y \in E$. According to a classical theorem of Rademacher, for any $\varphi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz the gradient $\nabla\varphi$ exists a.e. and the best constant in the previous inequality is $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}$.

An *unbounded Lipschitz domain* Ω in $\mathbb{R}^n$ is simply the domain above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, i.e. $\Omega = \{(x', x_n) : \varphi(x') < x_n\}$. Next, a *bounded* domain $\Omega \subset \mathbb{R}^n$, with connected boundary, is called *Lipschitz* if:

   i) there exist ψ_i, $1 \leq i \leq m$, rigid motions of the Euclidean space and a family of open, upright, circular cylinders $\{Z_i\}_{i=1}^m$ in $\mathbb{R}^n$ such that

$$\partial\Omega \subset \bigcup_{i=1}^m \psi_i^{-1}(Z_i). \tag{6.1}$$

   ii) for each i, there exists a Lipschitz function $\varphi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that if $t_i Z_i$ denotes the concentric dilation of Z_i by factor $t_i = 2\left(1 + \|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^{n-1})}^2\right)^{\frac{1}{2}}$, then

$$\psi_i(\Omega \cap \psi_i^{-1}(t_i Z_i)) = \{x = (x', x_n) : \varphi_i(x') < x_n\} \cap t_i Z_i,$$
$$\psi_i(\partial\Omega \cap \psi_i^{-1}(t_i Z_i)) = \{x = (x', x_n) : \varphi_i(x') = x_n\} \cap t_i Z_i, \tag{6.2}$$

   for each i.

In the sequel, we shall call $\{\mathcal{O}_i\}_{1 \leq i \leq m}$, $\mathcal{O}_i := \psi_i^{-1}(Z_i) \cap \partial\Omega$ an *atlas* for $\partial\Omega$, and we shall say that a constant depends on the Lipschitz character of Ω if its size is controlled in terms of m, the number of cylinders $\{Z_i\}_i$, the size of these cylinders and $\sup\{\|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^{n-1})} : 1 \leq i \leq m\}$.

It is sometimes useful to consider the special case of a *star-like* Lipschitz domain Ω. This implies the existence of a point $x^* \in \Omega$ and a Lipschitz function $\varphi : S^{n-1} \to \mathbb{R}$ with $\inf_{\omega \in S^{n-1}} \varphi(\omega) > 0$ such that, in polar coordinates (ρ, ω), the domain Ω has the parametric representation $\Omega = \{x = x^* + \omega\rho : \omega \in S^{n-1}, 0 < \rho < \varphi(\omega)\}$.

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain (unbounded, or star-like). The *radial maximal function* of a given function $u \in C_{\mathrm{loc}}^0(\Omega)$ is defined as

$$u_{\mathrm{rad}}(x) := \sup_{t>0} |u(x + te_n)|, \qquad x \in \bar{\Omega}, \tag{6.3}$$

if Ω is an unbounded Lipschitz domain, and

$$u_{\mathrm{rad}}(x) := \sup_{t>0} |u(e^{-t}x)|, \qquad x \in \bar{\Omega}, \tag{6.4}$$

if Ω is a star-like Lipschitz domain. These definitions are going to play a role in §11-§12.

Let now $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ we introduce

$$A_s^{p,q}(\Omega) := \{f \in D'(\Omega) : \exists g \in A_s^{p,q}(\mathbb{R}^n) \text{ such that } g|_\Omega = f\},$$
$$\|f\|_{A_s^{p,q}(\Omega)} := \inf\{\|g\|_{A_s^{p,q}(\mathbb{R}^n)} : g \in A_s^{p,q}(\mathbb{R}^n), g|_\Omega = f\}, \qquad f \in A_s^{p,q}(\Omega). \tag{6.5}$$

The convention we make in (6.5) is that either $A = F$ or $A = B$, corresponding to, respectively, the definition of Besov and Triebel-Lizorkin spaces in Ω. Hardy, Sobolev (or Bessel potential), Hölder and bmo spaces are defined analogously, namely

$$L_s^p(\Omega) := \{f \in D'(\Omega) : \exists g \in L_s^p(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}, \quad 1 < p < \infty, \ s \in \mathbb{R},$$
$$h^p(\Omega) := \{f \in D'(\Omega) : \exists g \in h^p(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}, \quad 0 < p \leq 1, \tag{6.6}$$
$$\mathrm{bmo}(\Omega) := \{f \in L_{\mathrm{loc}}^2(\Omega) : \exists g \in \mathrm{bmo}(\mathbb{R}^n) \text{ such that } g|_\Omega = f\},$$

equipped, in each case, with the natural, infimum-type, (quasi-)norms.

By (3.44)-(3.49), it follows that

$$C^s(\Omega) = B_s^{\infty,\infty}(\Omega), \quad 0 < s \notin \mathbb{Z}, \tag{6.7}$$

$$L^p(\Omega) = F_0^{p,2}(\Omega), \quad 1 < p < \infty, \tag{6.8}$$

$$L_s^p(\Omega) = F_s^{p,2}(\Omega), \quad 1 < p < \infty, \qquad s \in \mathbb{R}, \tag{6.9}$$

$$h^p(\Omega) = F_0^{p,2}(\Omega), \quad 0 < p \leq 1, \tag{6.10}$$

$$\mathrm{bmo}(\Omega) = F_0^{\infty,2}(\Omega), \tag{6.11}$$

where $C^s(\Omega)$ and $L^p(\Omega)$ are, respectively, the standard Hölder and Lebesgue spaces in Ω. It is immediate from these definitions that the restriction operator

$$\mathcal{R}_\Omega : S'(\mathbb{R}^n) \longrightarrow D'(\Omega), \qquad \mathcal{R}_\Omega f := f|_\Omega, \tag{6.12}$$

induces a linear and bounded operator in each of the following instances:

$$\mathcal{R}_\Omega : B_s^{p,q}(\mathbb{R}^n) \longrightarrow B_s^{p,q}(\Omega), \quad 0 < p, q \leq \infty, \ s \in \mathbb{R},$$
$$\mathcal{R}_\Omega : F_s^{p,q}(\mathbb{R}^n) \longrightarrow F_s^{p,q}(\Omega), \quad 0 < p < \infty, \ 0 < q \leq \infty, \ s \in \mathbb{R}. \tag{6.13}$$

We now pause to record a useful characterization of the local Hardy space $h^p(\Omega)$. First, we need some notation. Fix $\psi \in C_c^\infty(B(0,1))$ such that $\int_{B(0,1)} \psi(x)\,dx = 1$ and set $\psi_t(x) := t^{-n}\psi(x/t)$. Then the radial maximal function of a distribution u in Ω is defined as

$$u^+(x) := \sup_{0<t<\mathrm{dist}(x,\partial\Omega)} |(\psi_t * u)(x)|, \qquad x \in \Omega. \tag{6.14}$$

For $u \in D'(\Omega)$, $k \in \mathbb{N}_o$ and $x \in \mathbb{R}^n$ introduce

$$u_{k,\Omega}^*(x) := \sup\{|\langle u, \psi\rangle| : \psi \in \Psi_x\} \tag{6.15}$$

where the class Ψ_x consists of all functions $\psi \in C_c^\infty(\mathbb{R}^n)$ with the property that there exists $r = r_\psi > 0$ with $\mathrm{supp}\,\psi \subset B(x,r) \cap \Omega$ and $\|\partial^\gamma \psi\|_{L^\infty(\mathbb{R}^n)} \le r^{-n-|\gamma|}$ for each $\gamma \in \mathbb{N}_o^n$, $|\gamma| \le k$.

Theorem 6.1. *(cf. [66], [64]) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Fix ψ as above and define the radial maximal function as in (6.14). Then, for any $0 < p \le 1$ and any $u \in D'(\Omega)$*

$$u \in h^p(\Omega) \iff u^+ \in L^p(\Omega), \tag{6.16}$$

with equivalence of quasi-norms. Furthermore, if $k \in \mathbb{N}$ and $\frac{n}{n+k} < p \le 1$, then

$$\|u_{k,\Omega}^*\|_{L^p(\mathbb{R}^n)} \approx \|u^+\|_{L^p(\Omega)}. \tag{6.17}$$

In particular, a different choice of the function ψ affects the size of u^+ in $L^p(\Omega)$ only up to a fixed multiplicative constant. Finally, similar results are valid in the range $1 < p < \infty$ provided $h^p(\Omega)$ is replaced by $L^p(\Omega)$.

In analogy with the Hardy-based Sobolev spaces in $\mathbb{R}^n$ introduced in (3.10)-(3.11), for an open set $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}_o$ and $0 < p \le 1$, we set

$$h_k^p(\Omega) = \{u \in D'(\Omega) : \partial^\gamma u \in h^p(\Omega), \ \forall \gamma \in \mathbb{N}_o^n \text{ with } |\gamma| \le k\}, \tag{6.18}$$

equipped with the quasi-norm $\|u\|_{h_k^p(\Omega)} := \sum_{|\gamma|=k} \|\partial^\gamma u\|_{h^p(\Omega)}$, and

$$h_{-k}^p(\Omega) := \left\{u \in D'(\Omega) : u = \sum_{|\gamma| \le k} \partial^\gamma u_\gamma, \ u_\gamma \in h^p(\Omega) \ \forall \gamma \in \mathbb{N}_o^n \text{ with } |\gamma| \le k\right\} \tag{6.19}$$

equipped with $\|u\|_{h_{-k}^p(\Omega)} := \inf \sum_{|\gamma| \le k} \|u_\gamma\|_{h^p(\Omega)}$, where the infimum is taken over all representations of u.

Theorem 6.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $0 < p \le 1$ and that $k \in \mathbb{Z}$ is either ≤ 0, or else satisfies $k > n(1/p - 1)$. Then*

$$h_k^p(\Omega) = F_k^{p,2}(\Omega). \tag{6.20}$$

Proof. Assume first that $k < 0$ and note that the inclusion $h_k^p(\Omega) \hookrightarrow F_k^{p,2}(\Omega)$ is immediate from definitions, (3.55) and (6.10). To see the opposite inclusion fix $u \in F_k^{p,2}(\Omega)$, say $u = w|_\Omega$, $w \in F_k^{p,2}(\mathbb{R}^n)$. Since, by (3.51), w can be represented in the form $w = \sum_{|\gamma| \le -k} \partial^\gamma w_\gamma$, $w_\gamma \in h^p(\mathbb{R}^n)$, it follows that $u = \sum_{|\gamma| \le -k} \partial^\gamma(w_\gamma|_\Omega)$ and $w_\gamma|_\Omega \in h^p(\Omega)$. Consequently, $u \in h_k^p(\Omega)$, proving the right-to-left inclusion in (6.20). The case when $k \in \mathbb{N}$ satisfies $k > n(1/p - 1)$ is essentially due to A. Miyachi (cf. [66], [65]). More specifically, as on p. 80 of [65], let us temporarily introduce

$$\mathcal{W}_p^k(\Omega) := \left\{u \in h^p(\Omega) : \partial^\gamma u \in h^p(\Omega) \ \forall \gamma \in \mathbb{N}_o^n \text{ with } |\gamma| = k\right\} \tag{6.21}$$

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and observe that, by virtue of the last remark in §4 of [**65**],

$$\mathcal{W}_p^k(\Omega) = \mathcal{C}_p^k(\Omega), \tag{6.22}$$

where $\mathcal{C}_p^k(\Omega)$ is the space introduced by R. DeVore and R. Sharpley in §6 of [**28**]. Due to the extension results for the latter spaces proved in [**67**] (cf. Theorem 4/(ii) p. 1035 *loc. cit.*), it follows that $\mathcal{W}_p^k(\Omega) = \{u|_\Omega : u \in \mathcal{W}_p^k(\mathbb{R}^n)\}$ However, $\mathcal{W}_p^k(\mathbb{R}^n) = F_k^{p,2}(\mathbb{R}^n)$, thanks to (3.53) and (3.51) so that, altogether,

$$\mathcal{W}_p^k(\Omega) = F_k^{p,2}(\Omega). \tag{6.23}$$

Consequently, $h_k^p(\Omega) \hookrightarrow \mathcal{W}_p^k(\Omega) = F_k^{p,2}(\Omega)$, proving the left-to-right inclusion in (6.20). The opposite inclusion is a direct consequence of (3.55) and the fact that

$$\partial^\alpha : F_s^{p,q}(\Omega) \longrightarrow F_{s-|\alpha|}^{p,q}(\Omega) \tag{6.24}$$

is a bounded operator for each $\alpha \in \mathbb{N}_o^n$, $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. This completed the proof of the theorem. □

Returning to the mainstream discussion, let us single out two other types of function spaces which will play an important role later on. We continue to assume that Ω is an arbitrary open subset of $\mathbb{R}^n$. First, for $0 < p, q \leq \infty$, $s \in \mathbb{R}$, we set

$$\begin{aligned}
A_{s,0}^{p,q}(\Omega) &:= \{f \in A_s^{p,q}(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \\
\|f\|_{A_{s,0}^{p,q}(\Omega)} &:= \|f\|_{A_s^{p,q}(\mathbb{R}^n)}, \quad f \in A_{s,0}^{p,q}(\Omega),
\end{aligned} \tag{6.25}$$

where, as usual, either $A = F$ and $p < \infty$ or $A = B$. Thus, $B_{s,0}^{p,q}(\Omega)$, $F_{s,0}^{p,q}(\Omega)$ are closed subspaces of $B_{s,0}^{p,q}(\mathbb{R}^n)$ and $F_{s,0}^{p,q}(\mathbb{R}^n)$, respectively. In the same vein, we also define

$$\begin{aligned}
L_{s,0}^p(\Omega) &:= \{f \in L_s^p(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \quad 1 < p < \infty, \ s \in \mathbb{R}, \\
h_0^p(\Omega) &:= \{f \in h^p(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \quad 0 < p \leq 1, \\
\operatorname{bmo}_0(\Omega) &:= \{f \in \operatorname{bmo}(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\},
\end{aligned} \tag{6.26}$$

with the norms inherited from $L_s^p(\mathbb{R}^n)$, $h^p(\mathbb{R}^n)$ and $\operatorname{bmo}(\mathbb{R}^n)$ respectively. Second, for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, we introduce

$$\begin{aligned}
A_{s,z}^{p,q}(\Omega) &:= \{f \in D'(\Omega) : \exists g \in A_{s,0}^{p,q}(\Omega) \text{ such that } g|_\Omega = f\}, \\
\|f\|_{A_{s,z}^{p,q}(\Omega)} &:= \inf\{\|g\|_{A_s^{p,q}(\mathbb{R}^n)} : g \in A_{s,0}^{p,q}(\Omega), \ g|_\Omega = f\}, \quad f \in A_{s,z}^{p,q}(\Omega),
\end{aligned} \tag{6.27}$$

(where, as before, $A = F$ and $p < \infty$ or $A = B$) and, in keeping with earlier conventions,

$$L_{s,z}^p(\Omega) := F_{s,z}^{p,2}(\Omega) = \{f \in D'(\Omega) : \exists g \in L_{s,0}^p(\Omega) \text{ such that } g|_\Omega = f\}, \tag{6.28}$$

if $1 < p < \infty$, $s \in \mathbb{R}$, and

$$h_z^p(\Omega) := F_{0,z}^{p,2}(\Omega) = \{f \in D'(\Omega) : \exists g \in h_0^p(\Omega) \text{ such that } g|_\Omega = f\}, \text{ if } 0 < p \leq 1, \tag{6.29}$$

once again equipped with natural, infimum-type, (quasi-)norms. Finally,

$$C_z^\alpha(\Omega) := \{u \in C^\alpha(\overline{\Omega}) : u|_{\partial\Omega} = 0\}, \quad 0 < \alpha < 1, \tag{6.30}$$

$$\operatorname{bmo}_z(\Omega) := \{u|_\Omega : u \in \operatorname{bmo}(\mathbb{R}^n) \text{ with } \operatorname{supp} u \subseteq \overline{\Omega}\}. \tag{6.31}$$

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It follows that the restriction operator (6.12) induces linear, continuous mappings

$$\mathcal{R}_\Omega : B^{p,q}_{s,0}(\Omega) \longrightarrow B^{p,q}_{s,z}(\Omega),\ 0 < p, q \le +\infty,\ s \in \mathbb{R},$$

$$\mathcal{R}_\Omega : F^{p,q}_{s,0}(\Omega) \longrightarrow F^{p,q}_{s,z}(\Omega),\ 0 < p, q \le +\infty,\ s \in \mathbb{R}, \qquad (6.32)$$

$$\mathcal{R}_\Omega : L^{p}_{s,0}(\Omega) \longrightarrow L^{p}_{s,z}(\Omega),\ 1 < p < +\infty,\ s \in \mathbb{R},$$

$$\mathcal{R}_\Omega : h^{p}_{0}(\Omega) \longrightarrow h^{p}_{z}(\Omega),\ 0 < p \le 1.$$

In many instances it is important to establish whether there is a linear, bounded, extension operator, i.e., a right inverse for the various manifestations of the $\mathcal{R}_\Omega$. In the case of (6.13) when Ω is an arbitrary Lipschitz domain this problem has been solved in full generality by V. Rychkov (cf. [74]) who proved the following.

Theorem 6.3. ([74]) *Let $\Omega \subset \mathbb{R}^n$ be either a bounded Lipschitz domain, the exterior of a bounded Lipschitz domain, or an unbounded Lipschitz domain. Then there exists a linear, continuous operator $E_\Omega : D'(\Omega) \to S'(\mathbb{R}^n)$ such that whenever $0 < p, q \le +\infty$, $s \in \mathbb{R}^n$, then*

$$E_\Omega : A^{p,q}_s(\Omega) \longrightarrow A^{p,q}_s(\mathbb{R}^n)\ \text{boundedly, satisfying}\ \mathcal{R}_\Omega \circ E_\Omega f = f,\ \forall f \in A^{p,q}_s(\Omega), \qquad (6.33)$$

for $A = B$ or $A = F$, in the latter case assuming $p < \infty$.

This type of result has many forerunners. See [82] and [33] for the case of smooth domain. The theory on extension operators on Lipschitz domains was developed by different methods bringing corresponding restrictions on indices: Calderón's method [8], extended by E.M. Stein in [78] and then used by G.A. Kalyabin in [50] and T. Muramatu in [69], allowed to consider Banach space case only. A. Seeger [75], R.A. DeVore and R.C. Sharpley [29], and A. Miyachi [68] relied on approximation techniques and dealt with spaces consisting of locally integrable functions. Finally, results on Hardy spaces were obtained by A. Miyachi in [64]. An informative account of these and related matters can be found in [84] and [74].

In addition to identifications (6.7)–(6.10) we would like to discuss the classical Sobolev spaces

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) : \partial^\gamma f \in L^p(\Omega),\ \forall\, \gamma : |\gamma| \le k \right\}, \quad 1 < p < \infty,\ k \in \mathbb{N}, \qquad (6.34)$$

equipped with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|\gamma| \le k} \|\partial^\gamma f\|_{L^p(\Omega)}. \qquad (6.35)$$

It was proved in [8] that there exists a bounded linear extension operator

$$\mathcal{E}_k : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n)\ \text{such that}\ \mathcal{R}_\Omega \circ \mathcal{E}_k f = f,\ \forall f \in W^{k,p}(\Omega). \qquad (6.36)$$

In view of (3.4), (6.36) and (6.9) we then obtain

$$W^{k,p}(\Omega) = L^p_k(\Omega) = F^{p,2}_k(\Omega), \quad 1 < p < \infty, \qquad k \in \mathbb{N}. \qquad (6.37)$$

7 Complex interpolation of quasi-Banach spaces

The presentation in this subsection follows closely [47]. For a quasi-normed space $(X, \|\cdot\|_X)$, we denote by $\rho = \rho(X)$ its *modulus of concavity*, i.e. the smallest positive constant for which

$$\|x + y\|_X \le \rho(X)(\|x\|_X + \|y\|_X), \qquad x, y \in X. \qquad (7.1)$$

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Note that always $\rho(X) \geq 1$. We recall the Aoki-Rolewicz theorem, which asserts that X can be given an equivalent r-norm (where $2^{1/r-1} = \rho$) i.e. a quasi-norm which also satisfies the inequality:

$$\|x + y\| \leq (\|x\|^r + \|y\|^r)^{1/r}. \tag{7.2}$$

Cf., e.g., [**48**]). In general a quasi-norm need not be continuous but an r-norm is continuous. We shall assume however, throughout the paper that all quasi-norms considered are continuous. In fact, of course it would suffice to consider an r-norm for suitable r.

To set the stage for adapting Calderón's original complex method of interpolation to the setting of quasi-Banach spaces, we first review some basic results from the theory of analytic functions with values in quasi-Banach spaces as developed in [**85**], [**46**], [**45**].

Recall that if X is a topological vector space and U is an open subset of the complex plane then a map $f : U \to X$ is called *analytic* if given $z_0 \in U$ there exists $\eta > 0$ so that there is a power series expansion

$$f(z) = \sum_{j=0}^{\infty} (z - z_0)^n x_n, \quad x_n \in X, \text{ uniformly convergent for } |z - z_0| < \eta. \tag{7.3}$$

As explained in [**46**], in the context of quasi-Banach spaces, this is the most natural definition. Indeed, there are simple examples which show that complex differentiability leads to an unreasonably weaker concept of analyticity (see also [**85**] and [**3**] in this regard).

Proposition 7.1. *Suppose $0 < p \leq 1$ and that $m \in \mathbb{N}$ is such that $m > \frac{1}{p}$. Then there is a constant $C = C(m, p)$ so that if X is a p-normed quasi-Banach space and $f : \bar{D} \to X$ is a continuous function which is analytic on the unit disk $D := \{z : |z| < 1\}$ then for $z \in D$ we have $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$, and*

$$\|f^{(n)}(0)\|_X \leq C(m + n)! \sup_{z \in D} \|f(z)\|_X. \tag{7.4}$$

This is Theorem 6.1 of [**46**].

Proposition 7.2. *Let X be a quasi-Banach space and let U be an open subset of the complex plane. Let $f_n : U \to X$ be a sequence of analytic functions. If $\lim_{n \to \infty} f_n(z) = f(z)$ uniformly on compacta then f is also analytic.*

This follows from Theorem 6.3 of [**46**].

Proposition 7.3. *Suppose X is a quasi-Banach space and that U is an open subset of the complex plane. Let $f : U \to X$ be a locally bounded function. Suppose there is a weaker Hausdorff vector topology τ_0 on X which is locally p-convex for some $0 < p < 1$ and such that $f : U \to (X, \tau_0)$ is analytic. Then $f : U \to X$ is analytic.*

This is Theorem 3.3 of [**47**]. It shows that, many times in practice, the ambient space (within which the interpolation process is carried out) plays only a minor role in the setup. More specifically, assume that Y is a space of distributions in which a quasi-Banach space X is continuously embedded. Then, having an X-valued function analytic for the quasi-norm topology is basically the same as requiring analyticity for the weak topology (induced on X from Y).

We are now prepared to elaborate on the complex method of interpolation for pairs of quasi-Banach spaces. Consider a compatible couple (pair) of quasi-Banach spaces X_0, X_1, i.e. X_j, $j = 0, 1$, are continuously embedded into a larger topological vector space Y, and $X_0 \cap X_1$ is dense in X_j, $j = 0, 1$. Also, let U stand for the strip $\{z \in \mathbb{C} : 0 < \Re e\, z < 1\}$.

A family $\mathcal{F}$ of functions which map U into $X_0 + X_1$ is called *admissible* provided the following axioms are satisfied:

(i) $\mathcal{F}$ is a (complex) vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{F}}$ with respect to which it is complete (i.e. $\mathcal{F}$ is a quasi-Banach space);

(ii) the point-evaluation mappings $\mathrm{ev}_w : \mathcal{F} \to X_0 + X_1$, $w \in U$, defined by $\mathrm{ev}_w(f) := f(w)$ are continuous;

(iii) for any K compact subset of U there exists a positive constant C such that for any $w \in K$ and any $f \in \mathcal{F}$ with $f(w) = 0$, it then follows that the mapping $U \setminus \{w\} \ni z \mapsto f(z)/(z-w) \in X_0 + X_1$ extends to an element in $\mathcal{F}$ and

$$\left\| \frac{f(z)}{z-w} \right\|_{\mathcal{F}} \leq C\|f\|_{\mathcal{F}}. \tag{7.5}$$

These are the minimal requirements needed in order develop a reasonable interpolation theory at an abstract level. In practice, mimicking the Banach space theory, a common choice for $\mathcal{F}$, the class of admissible functions, is the space of bounded, analytic functions $f : U \to X_0 + X_1$, which is extended continuously to the closure of the strip such that the traces $t \mapsto f(j + it)$ are bounded continuous functions into X_j, $j = 0, 1$. We endow $\mathcal{F}$ with the quasi-norm

$$\|f\|_{\mathcal{F}} := \max \left\{ \sup_t \|f(it)\|_{X_0}, \sup_t \|f(1+it)\|_{X_1} \right\}. \tag{7.6}$$

However, there is an immediate problem that in general the evaluation maps ev_w are not necessarily bounded on $\mathcal{F}$ and so this class is not always admissible. In fact in the special case when $X_0 = X_1$ boundedness of the evaluation maps is equivalent to the validity of a form of the Maximum Modulus Principle. A quasi-Banach space X is *analytically convex* if there is a constant C such that for every polynomial $P : \mathbb{C} \to X$ we have $\|P(0)\|_X \leq C \max_{|z|=1} \|P(z)\|_X$. It is shown in [**45**] that if X is analytically convex it has an equivalent quasi-norm which is plurisubharmonic (i.e. we can insist that the constant C above can be taken to be 1). Let us also point out that being analytically convex is equivalent to the condition that

$$\max_{0 < \Re e\, z < 1} \|f(z)\|_X \leq C \max_{\Re e\, z = 0,1} \|f(z)\|_X, \tag{7.7}$$

for any analytic function $f : \{z \in \mathbb{C} : 0 < \Re e\, z < 1\} \to X$ which is continuous on the closed strip $\{z \in \mathbb{C} : 0 \leq \Re e\, z \leq 1\}$.

The relevance of the concept of analytic convexity in the current context suggests we take a more systematic look at it. Clearly, any Banach space is analytically convex. Other useful criteria for analytic convexity are summarized in the following result, proved in [**45**], [**22**].

Theorem 7.4. *For a quasi-Banach space $(X, \|\cdot\|_X)$ the following conditions are equivalent:*

(i) *X is analytically convex;*

(ii) *X has an equivalent quasi-norm $\|\cdot\|$ which is plurisubharmonic, i.e.*

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\|\, d\theta \geq \|x\|, \qquad \text{for any } x, y \in X;$$

(iii) *X has an equivalent quasi-norm $\|\cdot\|$ so that $\log \|\cdot\|$ is plurisubharmonic;*

(iv) *X has an equivalent quasi-norm $\|\cdot\|$ so that $\|\cdot\|^p$ is plurisubharmonic for some $0 < p < +\infty$;*

(v) *X has an equivalent quasi-norm $\|\cdot\|$ so that $\|\cdot\|^p$ is plurisubharmonic for each $0 < p < +\infty$;*

(vi) there exists C so that $\max\{\|f(z)\|_X : 0 < \Re e\, z < 1\} \leq C\max\{\|f(z)\|_X : \Re e\, z = 0, 1\}$ for any analytic function $f : U := \{z \in \mathbb{C} : 0 < \Re e\, z < 1\} \to X$ which extends in a continuous and bounded fashion on the closed strip $\bar{U}$.

Directly from definitions (or as a consequence of the equivalence $(i) \Leftrightarrow (vi)$ above) we have

Proposition 7.5. *If X is an analytically convex quasi-Banach space and Y is a closed subspace of X then Y is also analytically convex.*

Other examples of analytically convex quasi-Banach spaces can be manufactured by means of the following simple observation, which closely parallels Proposition 2.3 in [**22**].

Lemma 7.6. *Assume that (Ω, Σ, μ) is a measure space, $0 < p \leq +\infty$ and that $(X, \|\cdot\|_X)$ is an analytically convex quasi-Banach space. Then $L^p(\Omega, X)$, the space of X-valued functions which are p-th power integrable on Ω, is an analytically convex quasi-Banach space.*

Proof. Assume first that $0 < p < \infty$. Then, by Theorem 7.4, there exists an equivalent quasi-norm $\|\cdot\|$ on X so that $\|\cdot\|^p$ is plurisubharmonic. It follows that for any $f, g \in L^p(\Omega, X)$ and each fixed $\omega \in \Omega$, the function $u_\omega(z) := \|f(\omega) + zg(\omega)\|^p$ is subharmonic. Hence, so is $U(z) := \int_\Omega u_\omega(z)\, d\mu(\omega) = \int_\Omega \|f(\omega) + zg(\omega)\|^p\, d\mu(\omega) = \|f + zg\|^p_{L^p(\Omega, X)}$. Now, the desired conclusion follows from Theorem 7.4.

When $p = +\infty$, we simply write

$$\frac{1}{2\pi} \int_0^{2\pi} \sup_{\omega \in \Omega} \|f(\omega) + e^{i\theta}g(\omega)\|_X\, d\theta \geq \sup_{\omega \in \Omega} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(\omega) + e^{i\theta}g(\omega)\|_X\, d\theta\right)$$
$$\geq \sup_{\omega \in \Omega} \|f(\omega)\|_X. \tag{7.8}$$

The proof of the lemma is finished. $\qquad\qquad\square$

The relevance of this lemma for the applications we have in mind is brought forward by the following.

Proposition 7.7. *For any $s \in \mathbb{R}$, $0 < p, q \leq +\infty$, the spaces $F_s^{p,q}(\mathbb{R}^n)$ and $B_s^{p,q}(\mathbb{R}^n)$ are analytically convex. The same is true for their homogeneous versions and for the associated (homogeneous and inhomogeneous) discrete spaces.*

Proof. Let us deal with the claims made in the statement of the proposition for the inhomogeneous versions of the spaces in question. Thanks to Theorem 3.1, it suffices to work with sequence spaces. In this setting, observe that the assignments

$$f_s^{p,q} \ni \{\lambda_Q\}_{Q \in \mathcal{Q}_n^*} \mapsto f \in L^p\left(\mathbb{R}^n, \ell^q(\mathcal{Q}_n^*)\right)$$
$$f(x) := \left\{|Q|^{-s/n-1/2}\lambda_Q \chi_Q(x)\right\}_{Q \in \mathcal{Q}_n^*} \tag{7.9}$$

and

$$b_s^{p,q} \ni \{\lambda_Q\}_{Q \in \mathcal{Q}_n^*} \mapsto \{f_j\}_{j \in \mathbb{N}_o} \in \ell^q\left(\mathbb{N}_o, \ell^p(\mathcal{Q}_n^*)\right) \tag{7.10}$$

$$f_j(Q) := \begin{cases} |Q|^{-s/n-1/2+1/p}\lambda_Q & \text{if } Q \in \mathcal{Q}_n^* \text{ has } \ell(Q) = 2^{-j}, \\ 0 & \text{otherwise,} \end{cases} \tag{7.11}$$

are linear isomorphisms onto their images. Then the desired conclusion follows from Lemma 7.6 and Proposition 7.5. The case of the homogeneous versions of these spaces is analogous and this concludes the proof of the proposition. □

We now discuss several useful criteria for analytic convexity in the context of quasi-Banach *lattices* of functions. To set the stage, assume that (Ω, Σ, μ) is a σ−finite measure space and denote by L_0 the space of all complex-valued, μ-measurable functions on Ω. Then a *quasi-Banach function space* X on (Ω, Σ, μ), equipped with a quasi-norm $\| \cdot \|_X$ so that $(X, \| \cdot \|_X)$ is complete, is an order-ideal in the space L_0 if it contains a strictly positive function and if $f \in X$ and $g \in L_0$ with $|g| \leq |f|$ a.e. implies $g \in X$ with $\|g\|_X \leq \|f\|_X$. Going further, a quasi-Banach lattice of functions $(X, \| \cdot \|_X)$ is called *lattice r-convex* if

$$\left\| \left(\sum_{j=1}^{m} |f_j|^r \right)^{1/r} \right\|_X \leq \left(\sum_{j=1}^{m} \|f_j\|_X^r \right)^{1/r} \tag{7.12}$$

for any finite family $\{f_j\}_{1 \leq j \leq m}$ of functions from X (see, e.g., [43]; cf. also [53], Vol. II). This implies that the space

$$[X]^r := \left\{ f \text{ measurable} : |f|^{1/r} \in X \right\}, \quad \text{normed by} \quad \|f\|_{[X]^r} := \| |f|^{1/r} \|_X^r, \tag{7.13}$$

is a Banach function space, called the *r-convexification* of X (cf. also [53], Vol. II, pp. 53-54, at least if $r > 1$). The theorem below was first proved ([43], [45]), though in this particular form it is stated in [58].

Theorem 7.8. *Let X be a (complex) quasi-Banach lattice of functions and denote by κ its modulus of concavity. Then the following assertions are equivalent:*

(i) *X is analytically convex;*

(ii) *X is lattice r-convex for some $r > 0$;*

(iii) *X is lattice r-convex for each $0 < r < (1 + log_2\kappa)^{-1}$.*

Proof. This follows directly from Theorem 4.4 in [45] and Theorem 2.2 in [43] provided X satisfies an upper p-estimate with $p := (1 + \log_2\kappa)^{-1}$. That is, for some equivalent quasi-norm $\| \cdot \|$ and some constant $C > 0$,

$$\| |x_1| \vee ... \vee |x_n| \|^p \leq C \sum_{j=1}^{n} \|x_j\|^p \tag{7.14}$$

for any finite collection $x_1, ..., x_n \in X$. However, this is a simple consequence of the fact that in our case $|x_1| \vee ... \vee |x_n| \leq |x_1| + ... + |x_n|$ and the Aoki-Rolewicz theorem (recalled at the beginning of §7). □

Returning to the task of discussing the complex method for an interpolation couple of quasi-Banach spaces X_0, X_1, let us make the additional assumption that $X_0 + X_1$ is analytically convex. This entails

$$\sup \left\{ \|f(z)\|_{X_0+X_1} : 0 < \Re e\, z < 1 \right\} \leq C\|f\|_{\mathcal{F}}, \tag{7.15}$$

uniformly for $f \in \mathcal{F}$. With this in hand, all the aforementioned deficiencies of the complex method in the context of quasi-Banach spaces (such as the continuity of evaluation functions and the completeness of space $\mathcal{F}$) are easily corrected. We must thus define the *outer complex interpolation spaces* $X_\theta = [X_0, X_1]_\theta$ by $x \in X_\theta$ if and only if $x \in \mathcal{F}(\theta)$ and

$$\|x\|_\theta := \inf \left\{ \|f\|_{\mathcal{F}} : f(\theta) = x \right\}. \tag{7.16}$$

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It then follows that X_θ is a quasi-Banach space for $0 < \theta < 1$.

Let us note at this point that there is alternative choice for the class of admissible functions. We define $\mathcal{F}_0$ to be a subspace of $\mathcal{F}$ consisting of the closure those functions $f \in \mathcal{F}$ such that $f(w) \in X_0 \cap X_1$ for $w \in U$. We will use these this class to induce the *inner complex interpolation spaces*, $X_\theta^i = [X_0, X_1]_\theta^i$ by $x \in X_\theta$ if and only if $x \in \mathcal{F}_0(\theta)$ and

$$\|x\|_\theta := \inf\{\|f\|_{\mathcal{F}_0} : f(\theta) = x\}. \tag{7.17}$$

The inner spaces have the advantage that it is an immediate consequence of the definition that $X_0 \cap X_1$ is dense in each X_θ^i.

If X_0 and X_1 are Banach spaces the inner and outer complex methods yield exactly the same spaces (isometrically). But the argument for this depends essentially on the fact that X_0 and X_1 are Banach spaces. The idea of the proof is that if $f \in \mathcal{F}$ then $f_\epsilon(z) := f(z)e^{\epsilon z^2} \in \mathcal{F}_0$. To see the latter one computes

$$f_{\epsilon,n}(z) = \int_{-\infty}^{\infty} \varphi_n(t) f_\epsilon(z + it) dt \tag{7.18}$$

where $\varphi_n \in L^1(\mathbb{R})$ satisfies

$$\hat{\varphi}_n(t) = \int_{-\infty}^{\infty} \varphi_n(x)e^{-itx} dx = \left(1 - \frac{|t|}{n}\right)_+. \tag{7.19}$$

Then, with 'hat' denoting the Fourier transform on the real line,

$$\begin{aligned}
f_{\epsilon,n}(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}_n(t) \left(\int_{-\infty}^{\infty} f_\epsilon(z + is)e^{-ist} ds\right) dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}_n(t)e^{zt} \left(\int_{-\infty}^{\infty} f_\epsilon(z + is)e^{-(z+is)t} ds\right) dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}_n(t)e^{zt} G(t) dt, \tag{7.20}
\end{aligned}$$

where $G(t)$ is independent of z and can be shown to belong to $C^0(\mathbb{R}; X_0 \cap X_1)$. Thus, the functions $f_{\epsilon,n} \in \mathcal{F}_0$ and it follows that $f_\epsilon \in \mathcal{F}_0$.

Note that the above argument fails completely if X_0 and X_1 are quasi-Banach spaces because there is no corresponding integration theory. Thus, it is far from clear whether the inner and outer complex methods will always yield the same result in our setting. However, in special cases the inner and outer methods do yield the same result, as we will see below. Let us here note that complex interpolation of quasi-Banach spaces contained in an analytically convex ambient space (not necessarily $X_0 + X_1$) was first studied by Bernal and Cerdà in [5].

To state the next result, recall that, given two quasi-Banach lattices of functions $(X_j, \|\cdot\|_{X_j})$, $j = 0, 1$, the *Calderón product* $X_0^{1-\theta} X_1^\theta$, $0 < \theta < 1$, is

$$\begin{aligned}
X_0^{1-\theta} X_1^\theta &:= \{h \in L_0 : \exists f \in X_0, \ g \in X_1 \text{ such that } |h| \le |f|^{1-\theta}|g|^\theta\}, \\
\|f\|_{X_0^{1-\theta} X_1^\theta} &:= \inf\Big\{\|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^\theta : |f| \le |f_0|^{1-\theta}|f_1|^\theta, \ f_j \in X_j, \ j = 0, 1\Big\}. \tag{7.21}
\end{aligned}$$

A simple yet important feature for us here is that the Calderón product "commutes" with the process of convexification. More concretely, if X_0, X_1 are as above and, in addition, X_0, X_1 are also lattice r-convex for some $r > 0$, it is straightforward to check that

$$[X_0^{1-\theta} X_1^\theta]^r = ([X_0]^r)^{1-\theta}([X_1]^r)^\theta, \qquad \forall \theta \in (0, 1), \tag{7.22}$$

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in the sense of equivalence of quasi-norms.

It has been pointed out in [47] that the complex method described above gives the result predicted by the Calderón formula for nice pairs of function spaces. Let us record a specific result, building on earlier work in [40] and which has been proved in [47] for what we now call the outer method. To state it, recall that a Polish space is a topological space that is homeomorphic to some complete separable metric space.

Theorem 7.9. *Let Ω be a Polish space and let μ be a $\sigma-$finite Borel measure on Ω. Let X_0, X_1 be a pair of quasi-Banach function spaces on (Ω, μ). Suppose that both X_0 and X_1 are analytically convex and separable. Then $X_0 + X_1$ is analytically convex and, for each $\theta \in (0, 1)$,*

$$[X_0, X_1]_\theta = [X_0, X_1]_\theta^i = X_0^{1-\theta} X_1^\theta \tag{7.23}$$

in the sense of equivalence of quasi-norms.

Remark. As pointed out in [47], the hypothesis of separability in this case is equivalent to σ-order continuity. For a general quasi-normed space X, this property asserts that a non-negative, non-increasing sequence of functions in X which converges a.e. to zero also converges to zero in the quasi-norm topology of X (cf., e.g., [53], Vol. II). An equivalent reformulation is that if $g \in X$ and $|f_n| \leq |g|$ for all n and $f_n \to f$ a.e. then $\|f_n - f\|_X \to 0$. For us, it is of interest to also note a result, proved in Theorem 1.29 of [16], to the effect that

$$\begin{array}{c} \text{one of the lattices } X_0, X_1 \\ \text{is } \sigma\text{-order continuous} \end{array} \implies \begin{array}{c} X_0^{1-\theta} X_1^\theta \text{ is } \sigma\text{-order continuous} \\ \text{for each } \theta \in (0, 1). \end{array} \tag{7.24}$$

Proof. Let us briefly why the inner and outer methods agree here. It fact the argument in [47] which shows that if $f \in [X_0, X_1]_\theta$ then a nearly optimal choice for $F \in \mathcal{F}$ is of the form $F(z) = u|f_0|^{1-z}|f_1|^z$ where $f_0 \in X_0$, $f_1 \in X_1$ and $|u| = 1$ a.e. But we can select a sequence of Borel sets $E_n \uparrow \Omega$ so that $\chi_{E_n} f_0, \chi_{E_n} f_1 \in X_0 \cap X_1$ and consider $F_n(z) = \chi_{E_n} F(z)$. Thus $F_n \in \mathcal{F}_0$ and using order-continuity one sees that $\|F_n - F\|_{\mathcal{F}} \to 0$. Thus $F \in \mathcal{F}_0$. $\square$

Remark. For sequence spaces (which are the main applications we have in mind), Theorem 7.9 continues to hold in the case when just one of the two quasi-Banach lattices X_0, X_1 is separable. Indeed, in [47], the separability hypotheses on X_j, $j = 0, 1$, was used to ensure that if $f_0 \in X_0$ and $f_1 \in X_1$ then the function $z \mapsto |f_0|^{1-z}|f_1|^z$ is admissible (i.e. belongs to $\mathcal{F}$). In fact, the one property which is not immediate is its continuity on the closure of the strip $0 < \Re e \, z < 1$. Nonetheless, this issue can be handled as follows.

If $g := |f_0|^{1-\theta}|f_1|^\theta \in X_0^{1-\theta} X_1^\theta$ for $f_j \in X_j$ and $0 < \theta < 1$, then $g\chi_E = |f_0\chi_E|^{1-\theta}|f_1\chi_E|^\theta$ for any $E \subset \Omega$. In particular, if E is finite then, clearly, $z \mapsto |f_0\chi_E|^{1-z}|f_1\chi_E|^z$ is admissible. Thus, as in [47], $g\chi_E \in [X_0, X_1]_\theta$ and $\|g\chi_E\|_{[X_0,X_1]_\theta} \leq C\|g\chi_E\|_{X_0^{1-\theta}X_1^\theta}$. Consider now $E_n \nearrow \Omega$, a nested family of finite sets exhausting Ω (which can be arranged if the X_j's are sequence spaces). Replacing E by $E_j \setminus E_k$ and using the fact that, by (7.24), $g\chi_{E_n} \to g$ in $X_0^{1-\theta}X_1^\theta$, ultimately gives that $\{g\chi_{E_n}\}_n$ is Cauchy in $[X_0, X_1]_\theta$. The same argument further yields that $\{g\chi_{E_n}\}_n$ converges to g in $X_0 + X_1$. Hence, $g \in [X_0, X_1]_\theta$ and $\|g\|_{[X_0,X_1]_\theta} \leq C\|g\|_{X_0^{1-\theta}X_1^\theta}$. From this point on, one proceeds as in the proof of Theorem 3.4 in [47].

In the second part of this section we discuss some general interpolation results which are going to play an important role in future considerations.

Theorem 7.10. *Let X_i, Y_i, Z_i, $i = 0, 1$, be quasi-Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0, Z_1. Suppose that $Y_i \hookrightarrow Z_i$, $i = 0, 1$ and there exists a linear operator D*

such that $D : X_i \to Z_i$ *boundedly for* $i = 0, 1$. *Define the spaces*

$$X_i(D) := \{u \in X_i : Du \in Y_i\}, \quad i = 0, 1, \tag{7.25}$$

equipped with the graph norm, i.e. $\|u\|_{X_i(D)} := \|u\|_{X_i} + \|Du\|_{Y_i}$, $i = 0, 1$. *Finally, suppose that there exist continuous linear mappings* $G : Z_i \to X_i$ *and* $K : Z_i \to Y_i$ *with the property* $D \circ G = I + K$ *on the spaces* Z_i *for* $i = 0, 1$. *Then, for each* $0 < \theta < 1$ *and* $0 < q \le \infty$,

$$(X_0(D), X_1(D))_{\theta,q} = \{u \in (X_0, X_1)_{\theta,q} : Du \in (Y_0, Y_1)_{\theta,q}\}. \tag{7.26}$$

Furthermore, if the spaces $X_0 + X_1$ *and* $Y_0 + Y_1$ *are analytically convex, then*

$$[X_0(D), X_1(D)]_\theta = \{u \in [X_0, X_1]_\theta : Du \in [Y_0, Y_1]_\theta\}, \qquad \theta \in (0, 1). \tag{7.27}$$

Proof. For the real interpolation method it is convenient to work with K-functionals. The crux of the matter is establishing the following estimate:

$$K(t, a; X_0(D), X_1(D)) \approx K(t, a; X_0, X_1) + K(t, Da; Y_0, Y_1). \tag{7.28}$$

One direction is, of course, trivial. For the other one, given $a \in X_0 + X_1$, let

$$a = x_0 + x_1, \ \ x_i \in X_i, \ \text{ and } \ Da = y_0 + y_1, \ \ y_i \in Y_i, \quad i = 0, 1, \tag{7.29}$$

be nearly optimal splittings so that

$$\|x_0\|_{X_0} + t\|x_1\|_{X_1} \approx K(t, a, X_0, X_1) \tag{7.30}$$

and

$$\|y_0\|_{Y_0} + t\|y_1\|_{Y_1} \approx K(t, Da, Y_0, Y_1). \tag{7.31}$$

We then define a new splitting $a = x_0' + x_1'$, where

$$x_i' := x_i - GDx_i + Gy_i, \qquad i = 0, 1. \tag{7.32}$$

Then

$$\|x_i'\|_{X_i} \le C(\|x_i\|_{X_i} + \|y_i\|_{Y_i}), \qquad i = 0, 1. \tag{7.33}$$

Also

$$Dx_i' = -KDx_i + y_i + Ky_i, \qquad i = 0, 1, \tag{7.34}$$

so that

$$\|Dx_i'\|_{Y_i} \le C(\|x_i\|_{X_i} + \|y_i\|_{Y_i}), \qquad i = 0, 1, \tag{7.35}$$

since KD maps X_i boundedly into Y_i for $i = 0, 1$, and K maps Y_i boundedly to itself, $i = 0, 1$. The estimates (7.33)-(7.35) then justify the equivalence (7.28).

The identity (7.27), regarding complex interpolation, is due to J.-L. Lions and E. Magenes (cf. [**54**]) when all spaces involved are Banach. However, their argument goes through with minor modifications for quasi-Banach spaces given the analytic convexity assumptions made in this portion of our theorem. The only thing we need to check is that the space $X_0(D) + X_1(D)$ is analytically convex, so that the complex interpolation method outlined in the first part of this subsection applies to the couple $X_0(D)$, $X_1(D)$.

In order to justify this we first note that

$$X_0(D) + X_1(D) = (X_0 + X_1)(D) := \{u \in X_0 + X_1 : Du \in Y_0 + Y_1\}, \tag{7.36}$$

where the rightmost space is equipped with the natural graph norm. Indeed, (7.36) follows readily from the decompositions (7.29), (7.32). Thus, it suffices to prove that $(X_0 + X_1)(D)$ is analytically convex. To this end, let $f : U \to (X_0 + X_1)(D)$ be an analytic function which extends by continuity to $\bar{U}$. Since the inclusion $\iota : (X_0 + X_1)(D) \to X_0 + X_1$ is linear and bounded, f can also be regarded as an $(X_0 + X_1)$-valued analytic function in U, extendible by continuity to $\bar{U}$. Similarly, the operator $D : (X_0 + X_1)(D) \to Y_0 + Y_1$ is linear and bounded, thus Df is a $Y_0 + Y_1$-valued analytic function in U, which extends by continuity to $\bar{U}$. Consequently, given that $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex, we may write

$$
\max_{0 < \Re e\, z < 1} \|f(z)\|_{(X_0+X_1)(D)} \approx \max_{0 < \Re e\, z < 1} \|f(z)\|_{X_0+X_1} + \max_{0 < \Re e\, z < 1} \|Df(z)\|_{Y_0+Y_1}
$$

$$
\leq C \max_{\Re e\, z = 0,1} \|f(z)\|_{X_0+X_1} + C \max_{\Re e\, z = 0,1} \|Df(z)\|_{Y_0+Y_1}
$$

$$
\approx C \max_{\Re e\, z = 0,1} \|f(z)\|_{(X_0+X_1)(D)}, \tag{7.37}
$$

as desired. $\qquad\square$

We conclude this subsection with a simple, yet useful result, which is essentially folklore. First, we make a definition. Let X_0, X_1 and Y_0, Y_1 be two compatible pairs of quasi-Banach spaces. Call $\{Y_0, Y_1\}$ a *retract* of $\{X_0, X_1\}$ if there exist two bounded, linear operators $E : Y_i \to X_i$, $R : X_i \to Y_i$, $i = 0, 1$, such that $R \circ E = I$, the identity map, on each Y_i, $i = 0, 1$.

Lemma 7.11. *Assume that X_0, X_1 and Y_0, Y_1 are two compatible pairs of quasi-Banach spaces such that $\{Y_0, Y_1\}$ is a retract of $\{X_0, X_1\}$ (as before, the "extension-restriction" operators are denoted by E and R, respectively). Then for each $\theta \in (0, 1)$ and $0 < q \leq \infty$,*

$$
[Y_0, Y_1]_\theta = R\Big([X_0, X_1]_\theta\Big) \quad and \quad (Y_0, Y_1)_{\theta,q} = R\Big((X_0, X_1)_{\theta,q}\Big). \tag{7.38}
$$

In the case of the complex method, it is assumed that $X_0 + X_1$ is analytically convex.

As a corollary, we also have the following. Assume that (X_0, X_1) is a compatible pair of quasi-Banach spaces and that P is a common projection (i.e., a linear, bounded operator on X_i, $i = 0, 1$, such that $P^2 = P$). Then the real and complex interpolation brackets commute with the action of P, i.e.

$$
[PX_0, PX_1]_\theta = P\Big([X_0, X_1]_\theta\Big) \quad and \quad (PX_0, PX_1)_{\theta,q} = P\Big((X_0, X_1)_{\theta,q}\Big), \tag{7.39}
$$

for each $\theta \in (0, 1)$ and $0 < q \leq \infty$. In the case of the complex method, it is assumed that $X_0 + X_1$ is analytically convex.

Remark. (*i*) Generally speaking, given two quasi-normed spaces X, Y and a linear, bounded operator $T : X \to Y$, by TX we shall denote its image equipped with the quasi-norm

$$
\|y\|_{TX} := \inf\{\|x\|_X : x \in X \text{ such that } y = Tx\}, \qquad y \in TX. \tag{7.40}
$$

In particular, this is the sense in which (7.38) and (7.39) should be understood.

(*ii*) The portion of Lemma 7.11 referring to real interpolation remains valid when the spaces in question are quasi-normed Abelian groups (in which case, the operators involved are assumed to be group morphisms).

Proof of Lemma 7.11. The first order of business is to show that $Y_0 + Y_1$ is analytically convex (hence, justifying the use of the complex method of interpolation for the pair Y_0, Y_1). One way to see this is by

observing that E maps $Y_0 + Y_1$ isomorphically onto $E(Y_0 + Y_1)$ which, given that this operator has a left inverse, is a closed subspace of the analytically convex space $X_0 + X_1$. Hence, $Y_0 + Y_1$ is also analytically convex.

The remainder of the proof follows a well-known path. We, nonetheless, include the details for the convenience of the reader. Fix $\theta \in (0, 1)$ and set $X_\theta = [X_0, X_1]_\theta$, $Y_\theta = [Y_0, Y_1]_\theta$. By the interpolation property for bounded linear operators, R maps X_θ to Y_θ, i.e., $R(X_\theta) \subseteq Y_\theta$. To justify the opposite inclusion, note that E takes Y_θ into X_θ which is further mapped by R into $R(X_\theta)$. Since the composition of these two applications acts as the identity operator, we may conclude that $Y_\theta \subseteq R(X_\theta)$, as desired. The proof in the case of the real interpolation method is virtually the same and this completes the proof of the first part of the lemma.

Turning to the second part of our lemma, we note that $\{PX_0, PX_1\}$ is a retract of $\{X_0, X_1\}$ (taking E to be the inclusion and R the given projection). Thus, (7.39) is a corollary of what we have proved so far. $\qquad\square$

8 Perturbation results on complex interpolation scales

Here we discuss a very useful result which essentially asserts that, on a complex interpolation scales of quasi-Banach spaces, the property of being invertible is stable and the inverses are compatible. The Banach space version can be found in [**11**], [**77**], [**2**], [**76**], [**87**]. The theorem below builds on the work in [**47**], where other related results can be found.

Theorem 8.1. *Let X_0, X_1 and Y_0, Y_1 be two compatible couples of quasi-Banach spaces and assume that $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex. Also, consider a bounded, linear operator $T : X_j \to Y_j$, $j = 0, 1$. If $X_\theta := [X_0, X_1]_\theta$ and $Y_\theta := [Y_0, Y_1]_\theta$, or if $X_\theta := [X_0, X_1]_\theta^i$ and $Y_\theta := [Y_0, Y_1]_\theta^i$, then for each $\theta \in (0, 1)$, then T induces a bounded linear operator*

$$T_\theta : X_\theta \longrightarrow Y_\theta, \qquad \theta \in (0, 1), \tag{8.1}$$

in a natural fashion. Moreover,

$$\|T_\theta\|_{X_\theta \to Y_\theta} \leq \|T\|_{X_0 \to X_0}^{1-\theta} \|T\|_{X_1 \to X_1}^\theta \qquad \theta \in (0, 1). \tag{8.2}$$

Assume next that there exists $\theta_o \in (0, 1)$ such that T_{θ_o} is an isomorphism. Then there exists $\varepsilon > 0$ such that T_θ continues to be isomorphism whenever $|\theta - \theta_o| < \varepsilon$.

Furthermore, if I is any open subinterval of $(0, 1)$ with the property that T_θ^{-1} exists for every $\theta \in I$, then T_θ^{-1} agrees with $T_{\theta'}^{-1}$ on $Y_\theta \cap Y_{\theta'}$ for any $\theta, \theta' \in I$.

Proof. The interpolation property (8.1)-(8.2) along with the stability of the quality of being Fredholm or invertible for linear operators on complex interpolation scales of quasi-Banach spaces have already been established in [**47**]. Here we focus on the compatibility of inverses, stated in the last part of the theorem. For simplicity of notation, we shall assume that $X_j = Y_j$ for $j = 0, 1$. The proof in the general case follows analogously.

To get started, note that it suffices to show that if $\theta_o \in (0, 1)$ is such that T_θ is an isomorphism whenever $|\theta - \theta_o| < \varepsilon$, then T_θ^{-1} agrees with $T_{\theta'}^{-1}$ on $Y_\theta \cap Y_{\theta'}$ for any $\theta, \theta' \in (\theta_o - \varepsilon, \theta_o + \varepsilon)$.

Note that if T_{θ_o} is invertible then so is T_z for z in a neighborhood of θ_o in the strip U and we denote by T_z^{-1} its inverse. Now, if F is an arbitrary admissible function, there exists $G_1 \in \mathcal{F}$ satisfying the properties

$$T[G_1(\theta_o)] = F(\theta_o) \quad \text{and} \quad \|G_1\|_\mathcal{F} \leq 2\|F\|_\mathcal{F}\|T_{\theta_o}^{-1}\|. \tag{8.3}$$

Therefore, thanks to the axiom (iii) in the first part of §7, F can be represented as $F = TG_1 + \omega F_1$, where $F_1 \in \mathcal{F}$ and ω is a suitable analytic function with $\omega(\theta_o) = 0$ and $|\omega| < 1$ on the boundary of the strip U (for instance, $\omega(z) := \frac{1}{3}(z - \theta_o)\, e^{z^2}$ will do). Also, by (7.5),

$$\|F_1\|_{\mathcal{F}} \leq \kappa \|F\|_{\mathcal{F}}, \qquad \kappa := \rho(\mathcal{F})(2\|T\|\|T_{\theta_o}^{-1}\| + 1). \tag{8.4}$$

Iterating this process, we obtain

$$F_i = TG_{i+1} + \omega F_{i+1}, \tag{8.5}$$

where, for $i = 1, 2, ...,$

$$\|F_{i+1}\|_{\mathcal{F}} \leq \kappa^{i+1}\|F\|_{\mathcal{F}} \quad \text{and} \quad \|G_{i+1}\|_{\mathcal{F}} \leq 2\|F_i\|_{\mathcal{F}}\|T_{\theta_o}^{-1}\| \leq 2\kappa^i\|F\|_{\mathcal{F}}\|T_{\theta_o}^{-1}\|. \tag{8.6}$$

Thus,

$$F = T\Big[\sum_{k=1}^{n} \omega^{k-1} G_k\Big] + \omega^n F_n, \quad \text{for every } n \in \mathbb{N}. \tag{8.7}$$

Granted the estimates (8.6), it is not too difficult to see that there exist V, neighborhood of θ_o such that $\sum_{k=1}^{\infty} \omega^{k-1} G_k$ converges to a function G uniformly on compacta and $\omega^n F_n$ converges to 0. By Proposition 7.2, the function G is analytic as an $(X_0 + X_1)$-valued function and satisfies $T[G(z)] = F(z)$, $G(z) \in X_z$ for every $z \in V$. In particular, $T_z^{-1}[F(z)] = G(z)$ is analytic for $z \in V$.

Next, recall that $X_0 \cap X_1$ is dense in X_z for all z, and fix some $x \in X_0 \cap X_1$. From the above reasoning it follows that there exists V neighborhood of θ_o in U such that $T_z^{-1}x$, viewed as a mapping $V \ni z \mapsto T_z^{-1}[F_o(z)] \in X_0 + X_1$, where $F_o \in \mathcal{F}$ is the constant function $F_o(z) := x$, is analytic. Since this function is also independent of the imaginary part of z, we may conclude that it is a constant function. Thus, $T_z^{-1}x \in X_0 + X_1$ is independent of $z \in V$, as desired. $\square$

A few comments are in order. First, the compatibility condition for inverses is related to the so-called *global uniqueness of resolvent* condition. See [88] where a spectral invariance theorem is proved for operators satisfying global uniqueness of resolvent condition relative to the real method of interpolation.

Second, it is well-known that there exist linear operators T mapping $L^p(\mathbb{R})$ boundedly into itself for every $p \in (1, \infty)$ which happen to be invertible for two distinct values of p, say p_0 and p_1, and yet the respective inverses do not agree on $L^{p_0}(\mathbb{R}) \cap L^{p_1}(\mathbb{R})$. On the positive side, it is very easy to check that, in the context of Theorem 8.1, all inverses act in a coherent fashion (provided they exist) if, e.g., $X_0 \hookrightarrow X_1$ and $Y_0 \hookrightarrow Y_1$.

The fact that the isomorphism property for T_θ is stable under small perturbations can be extended to the property of being Fredholm, but some technicalities intervene. Unfortunately the presentation of this result in [47] is somewhat mangled so we take this opportunity to correct it. We first note that it follows directly from Lemma 2.8 of [47] that we have:

Theorem 8.2. *Under hypotheses of Theorem 8.1, if T_{θ_o} is surjective and has finite-dimensional kernel then there exists $\epsilon > 0$ so that $\dim \ker T_\theta$ is constant for $|\theta - \theta_o| < \epsilon$.*

To extend this to Fredholm operators a device is used in [47] (discussion prior to Theorem 2.9) where the so-called intersection property is introduced. Unfortunately, there is a slight misstatement in that $\tilde{Z}$ should not be required to be closed in Z. However, it would be simpler, in retrospect, to define the intersection property by requiring only that $\tilde{Z}$ is a common dense subspace of each X_w and that $\mathcal{F}$ contains the constant $\tilde{Z}$−valued functions; the proof of Theorem 2.9 would go through verbatim. In the proof of Theorem 2.9 of [47], $\mathcal{F}_E$ should be replaced by $\mathcal{G}_E$.

Let us show how this works for the case of interpolation of two spaces, where we take $\tilde{Z} = X_0 \cap X_1$:

Theorem 8.3. *Retain the same hypotheses as in Theorem 8.1 and assume that $Y_0 \cap Y_1$ is dense in each Y_θ for $0 < \theta < 1$ (which is automatic for the case of inner complex interpolation). Then if T_{θ_o} is Fredholm there exists $\epsilon > 0$ so that T_θ is Fredholm for $|\theta - \theta_0| < \epsilon$ and the index is constant.*

Proof. Since $Y_0 \cap Y_1$ is dense in X_{θ_o} we may find a finite-dimensional subspace E of $Y_0 \cap Y_1$ such that $E \oplus T_{\theta_o}(X_{\theta_o}) = Y_{\theta_o}$. Consider the pair $(X_0 \oplus E, X_1 \oplus E)$ and the map $S : X_j \oplus E \to Y_j$ defined by $S(x, e) = Tx + e$. Then forming the same (inner or outer) interpolation spaces Z_θ for the new pair one shows that $Z_\theta = X_\theta \oplus E$, $S_\theta(x, e) = T_\theta x + e$, and S_{θ_o} is surjective. Applying Theorem 8.2 to this operator gives that $\dim \ker S_\theta$ is locally constant. However $\operatorname{ind} T_\theta = \operatorname{ind} S_\theta - \dim E = \dim \ker S_\theta - \dim E$ is locally constant. $\square$

The motivation for studying this type of question stems from their usefulness in the context of PDE's (see, e.g., [**57**], [**61**], [**62**], [**63**]). In his work on the oblique derivative problem [**7**], A.P. Calderón proved the following result. If (X, μ) is a measure space and $T : L^p(X, \mu) \to L^p(X, \mu)$ is a bounded operator for $1 < p < \infty$ which happens to be invertible when $p = 2$ then T is also invertible when $2 - \varepsilon < p < 2 + \varepsilon$, for some small $\varepsilon > 0$. Independently (and considerably earlier) I.Ya. Šneĭberg has proved in [**77**] a much more general result (a spectral continuity theorem on complex interpolation scales of Banach spaces to be more exact –later extended in [**47**] the case of quasi-Banach spaces), but Calderón's method, besides being relatively short and elementary, has the advantage of providing a Neumann series expansion for the inverse from which, although not explicitly stated, it is possible to read off the compatibility of inverses. More specifically, there exists some small $\varepsilon > 0$ such that for each $p, q \in (2 - \varepsilon, 2 + \varepsilon)$, the inverse T^{-1} considered on the space $L^p(X, \mu)$ is compatible with T^{-1} considered on $L^q(X, \mu)$ when both operators are restricted to $L^p(X, \mu) \cap L^q(X, \mu)$. As pointed out in [**71**], the latter condition turns out to be very useful for applications.

There are several other important and nontrivial instances where compatibility results of the kind described above play a basic role. One such example is as follows. From the work of Dahlberg and Kenig [**20**] it is known that given $\Omega \subset \mathbb{R}^n$, the unbounded domain above the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$, the Dirichlet problem for the Laplacian

$$\text{(DBVP)} \quad \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ M(u) \in L^p(\partial\Omega), \\ u\Big|_{\partial\Omega} = f \in L^p(\partial\Omega), \end{cases} \tag{8.8}$$

is well-posed for each $2 - \varepsilon < p < \infty$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$. Above, M stands for the non-tangential maximal operator defined by

$$Mu(x) := \sup\{|u(y)| : y \in \Omega, |x - y| < 2\operatorname{dist}(y, \partial\Omega)\}, \qquad x \in \partial\Omega. \tag{8.9}$$

and the trace is taken in the sense of nontangential convergence to the boundary, i.e.,

$$u\Big|_{\partial\Omega}(x) := \lim_{\substack{\Omega \ni y \to x \\ |x-y| < 2\operatorname{dist}(y, \partial\Omega)}} u(y), \qquad x \in \partial\Omega. \tag{8.10}$$

A natural question in this context is as follows. Assume that $2 - \varepsilon < p, q < \infty$ and that u is harmonic in Ω with $M(u) \in L^p(\partial\Omega)$. Does it follow that

$$u\Big|_{\partial\Omega} \in L^q(\partial\Omega) \iff M(u) \in L^q(\partial\Omega)\,? \tag{8.11}$$

In other words, in the well-posedness range, can one read the size of a harmonic function (expressed in terms of the nontangential maximal operator) off the size of its trace? The answer turns out to be

"yes" given that, in the approach in [20], the solution is explicitly expressed in terms of its boundary trace via a formula which involves the inverse of a certain boundary integral operator. In order to be more specific, we need more notation. Let ω_n be the area of the unit sphere in $\mathbb{R}^n$ and denote by ν the outward unit normal to Ω, which is well-defined with respect to the boundary surface measure σ at a.e. point on $\partial\Omega$. Then the harmonic double layer potential operator and its (principal-value) boundary version are, respectively, defined by

$$\mathcal{D}f(x) \quad := \quad \frac{1}{\omega_n} \int_{\partial\Omega} \frac{\langle \nu(y), y-x \rangle}{|x-y|^n} f(y)\, d\sigma(y), \qquad x \in \Omega, \tag{8.12}$$

$$\mathcal{K}f(x) \quad := \quad \lim_{\varepsilon \to 0^+} \frac{1}{\omega_n} \int_{\substack{y \in \partial\Omega \\ |x-y| > \varepsilon}} \frac{\langle \nu(y), y-x \rangle}{|x-y|^n} f(y)\, d\sigma(y), \qquad x \in \partial\Omega. \tag{8.13}$$

From the work in [15] and [86], it is known that

$$\|M(\mathcal{D}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \qquad \mathcal{D}f\Big|_{\partial\Omega} = (\tfrac{1}{2}I + \mathcal{K})f, \tag{8.14}$$

for each $f \in L^p(\partial\Omega)$, $1 < p < \infty$. Hence, a solution to (8.8) is given by

$$u(x) = \mathcal{D}\Big((\tfrac{1}{2}I + \mathcal{K})^{-1}f\Big)(x), \qquad x \in \Omega, \tag{8.15}$$

whenever the inverse $(\tfrac{1}{2}I + \mathcal{K})^{-1}$ exists in $L^p(\partial\Omega)$. The major advance in [20] was the proof of the fact that there exists $\varepsilon > 0$ such that

$$\tfrac{1}{2}I + \mathcal{K} : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \text{ is an isomorphism whenever } 2 - \varepsilon < p < \infty. \tag{8.16}$$

In this context, proving (8.11) reduces to checking the compatibility of $(\tfrac{1}{2}I + \mathcal{K})^{-1}$ considered on $L^p(\partial\Omega)$, and on $L^q(\partial\Omega)$, respectively, which is an issue addressed by Theorem 8.1.

The example just discussed involves only Banach spaces, but there are situations where working with quasi-Banach spaces is a necessity naturally dictated by the problem. This is the case for the Neumann problem

$$\text{(NBVP)} \quad \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ M(\nabla u) \in L^p(\partial\Omega), \\ \partial_\nu u = g \in H^p_{at}(\partial\Omega), \end{cases} \tag{8.17}$$

where ∂_ν denotes the normal derivative, $(n-1)/n < p \leq 1$ and $H^p_{at}(\partial\Omega)$ stands for the Hardy space on the Lipschitz hypersurface $\partial\Omega$. Much as with their flat-space counterpart, the Hardy paces $H^p_{at}(\partial\Omega)$ are only quasi-Banach when $p < 1$.

It turns out that the solvability of (8.17) hinges on the ability to invert the operator $-\tfrac{1}{2}I + \mathcal{K}^t$ on $H^p_{at}(\partial\Omega)$, where $\mathcal{K}^t$ is the formal transpose of (8.13). In this connection, it has been shown in [6], [47], that there exists $\varepsilon > 0$ such that

$$-\tfrac{1}{2}I + \mathcal{K}^t : H^p_{at}(\partial\Omega) \longrightarrow H^p_{at}(\partial\Omega) \text{ is an isomorphism whenever } 1 - \varepsilon < p \leq 1. \tag{8.18}$$

noindent The case $p = 1$ is due to Dahlberg-Kenig [20], and the approach in [47] was to use the stability of the property of linear operators of being invertible on complex interpolation scales of quasi-Banach

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spaces. An argument akin to that in the case of the Dirichlet problem then yields the following result. If $1 - \varepsilon < p, q \leq 1$ and u is harmonic in Ω with $M(\nabla u) \in L^p(\partial\Omega)$, then

$$\partial_\nu u \in H^q_{at}(\partial\Omega) \iff M(\nabla u) \in L^q(\partial\Omega). \tag{8.19}$$

There are many other PDE's amenable to this sort of analysis, including systems and even parabolic PDE's.

Finally, we would like to point out that a number of variations on the themes considered in the first part of this section are possible. For example, it is possible to incorporate an analytic dependence on the scale parameter in the operator T. Another natural issue is whether results of type discussed above are valid for the *real method of interpolation*. Progress in this direction has been recently made in [**60**].

9 Complex interpolation of Besov and Triebel-Lizorkin spaces: full scale results

The first result of this section is an extension of Theorem 5.1 to the full range of indices for which the Besov and Triebel-Lizorkin spaces are defined. More specifically, we have:

Theorem 9.1. *Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$ and $0 < q_0, q_1 \leq \infty$. Then (5.6) continues to hold in this setting where, as before, $0 < \theta < 1$, $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.*

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ and either $p_0 + q_0 < \infty$ or $p_1 + q_1 < \infty$ then (5.7) is also valid in the current setting where, as before, $0 < \theta < 1$, $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Similar results are valid for the homogeneous Besov and Triebel-Lizorkin spaces, i.e.,

$$\left[\dot{F}^{p_0,q_0}_{s_0}(\mathbb{R}^n), \dot{F}^{p_1,q_1}_{s_1}(\mathbb{R}^n)\right]_\theta = \dot{F}^{p,q}_s(\mathbb{R}^n). \tag{9.1}$$

$$\left[\dot{B}^{p_0,q_0}_{s_0}(\mathbb{R}^n), \dot{B}^{p_1,q_1}_{s_1}(\mathbb{R}^n)\right]_\theta = \dot{B}^{p,q}_s(\mathbb{R}^n), \tag{9.2}$$

under the same assumptions on the indices as before. Finally, analogous results are valid for the discrete scales of spaces.

Before presenting the proof, a few comments are in order here. First, this result seems new only when $\min\{p_0, q_0, p_1, q_1\} < 1$; for the other case see, e.g., Corollary 8.3 in [**35**] and Theorem 6.4.5 in [**4**].

Second, a result which formally resembles ours has been proved in §2.4.7 of [**82**]. Nonetheless, it should be pointed out that the complex method utilized there is different from ours and, more importantly, does not seem suited for the applications we have in mind. In particular, it is not known whether it has the so-called interpolation property (i.e., preservation of the boundedness of linear operators); see the comment at the beginning of §2.4.8 in [**82**].

Third, (9.1) contains the complex interpolation of Hardy spaces, i.e. for each $0 < \theta < 1$ and $0 < p_0, p_1 < +\infty$,

$$[H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)]_\theta = H^p(\mathbb{R}^n), \quad \text{where } 1/p := (1 - \theta)/p_0 + \theta/p_1. \tag{9.3}$$

It also contains complex interpolation between Hardy spaces and BMO, i.e.

$$[H^{p_0}(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n)]_\theta = H^p(\mathbb{R}^n), \quad 0 < \theta < 1, \ 0 < p_0 < +\infty, \ 1/p := (1 - \theta)/p_0. \tag{9.4}$$

The same formulas, but for *different* methods of complex interpolation, have been obtained in [**10**], [**41**], [**18**], [**40**]. Clearly, (9.1)-(9.2) contain several other particular cases of independent interest; we leave their formulation to the interested reader.

As a preamble to the proof of Theorem 9.1, we discuss a couple of auxiliary results.

Lemma 9.2. *For each $s \in \mathbb{R}$, $0 < p, q \leq +\infty$ and $0 < r < \min\{p, q\}$, the spaces $\dot{f}_s^{p,q}$ and $\dot{b}_s^{p,q}$ are lattice r-convex. Moreover,*

$$[\dot{f}_s^{p,q}]^r = \dot{f}_{s'}^{p',q'}, \qquad [\dot{b}_s^{p,q}]^r = \dot{b}_{s'}^{p',q'}, \tag{9.5}$$

where the indices are related by

$$p' = p/r, \quad q' = q/r, \quad s' = r(s + n/2) - n/2. \tag{9.6}$$

Similar results are valid for the inhomogeneous sequence spaces.

Proof. It is trivial to check that if $r > 0$ then $\{|\lambda_Q|^{1/r}\}_Q \in \dot{f}_s^{p,q} \Longleftrightarrow \{\lambda_Q\}_Q \in \dot{f}_{s'}^{p',q'}$, where the primed indices are as in (9.6). The claims made about the scale $\dot{f}_s^{p,q}$ are clear from this. The case of $\dot{b}_s^{p,q}$ is virtually identical. □

Note that we can also use the above lemma together with Theorem 7.8 in order to give an alternative proof of the fact that the sequence spaces $\dot{f}_s^{p,q}$, $\dot{b}_s^{p,q}$ are analytically convex.

Proposition 9.3. *For $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 \leq +\infty$, $0 < \theta < 1$, $1/p := (1 - \theta)/p_0 + \theta/p_1$, $1/q := (1 - \theta)/q_0 + \theta/q_1$ and $s := (1 - \theta)s_0 + \theta s_1$, there holds*

$$\dot{f}_s^{p,q} = \left(\dot{f}_{s_0}^{p_0,q_0}\right)^{1-\theta} \left(\dot{f}_{s_1}^{p_1,q_1}\right)^\theta. \tag{9.7}$$

If, in addition, $s_0 \neq s_1$, then also

$$\dot{b}_s^{p,q} = \left(\dot{b}_{s_0}^{p_0,q_0}\right)^{1-\theta} \left(\dot{b}_{s_1}^{p_1,q_1}\right)^\theta. \tag{9.8}$$

Finally, similar results are valid for the inhomogeneous sequence spaces.

Proof. The Calderón product on the $\dot{f}_s^{p,q}$ scale has been computed in Theorem 8.2 of [**35**]. However, a proof of (9.8) does not seem to be readily available in the literature; we include one here.

Our first observation is that (9.8) is valid if, in addition, $p_0, p_1, q_0, q_1 \geq 1$, in which case all spaces involved are actually Banach. Indeed, in this situation, (9.8) is a consequence of Calderón's formula (allowing one to identify the Calderón's product with the intermediate spaces obtained via complex interpolation; cf. [**9**]) and the fact that, under the current assumptions on the indices (cf. Theorem 5.2),

$$\dot{B}_s^{p,q}(\mathbb{R}^n) = \left[\dot{B}_{s_0}^{p_0,q_0}(\mathbb{R}^n), \dot{B}_{s_1}^{p_1,q_1}(\mathbb{R}^n)\right]_\theta. \tag{9.9}$$

Turning now to the general case, all we need to do is to utilize (7.22), for some $r > 0$ sufficiently small, in order to reduce matters to the Banach case (i.e. when all integrability indices are ≥ 1). Since this has been treated before, and since convexification commutes with the Calderón product, the desired conclusion follows easily. The proof of the proposition is therefore finished. □

We are now ready to present the

Proof of Theorem 9.1. Consider the case of the Triebel-Lizorkin scale. By virtue of Theorem 3.1, $F_{\alpha_0}^{p_0,q_0}(\mathbb{R}^n)$ and $F_{\alpha_1}^{p_1,q_1}(\mathbb{R}^n)$ have a common unconditional basis, namely (3.33). This is then an unconditional basis in the sum space, $F_{\alpha_0}^{p_0,q_0}(\mathbb{R}^n) + F_{\alpha_1}^{p_1,q_1}(\mathbb{R}^n)$ and the wavelet transform, i.e., the map associating to each distribution its sequence of wavelet coefficients defined as in (3.34), is an isomorphism between

$F_{\alpha_0}^{p_0,q_0}(\mathbb{R}^n) + F_{\alpha_1}^{p_1,q_1}(\mathbb{R}^n)$ and $f_{\alpha_0}^{p_0,q_0} + f_{\alpha_1}^{p_1,q_1}$, and between $F_{\alpha_i}^{p_i,q_i}(\mathbb{R}^n)$ and $f_{\alpha_i}^{p_i,q_i}$ for $\alpha_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$, $i = 0,1$. Thus, matters can be reduced to working with sequence spaces.

Next, from Proposition 7.7 and the discussion right below (3.24) we know that

$$f_\alpha^{p,q} \text{ is analytically convex if } 0 < p,q \le \infty, \ \alpha \in \mathbb{R}, \ \text{ and separable if } p + q < +\infty. \quad (9.10)$$

In concert with Theorem 7.9 and the remarks following it, this gives that, for any set of indices as in the statement of the theorem,

$$f_{\alpha_0}^{p_0,q_0} + f_{\alpha_1}^{p_1,q_1} \text{ is analytically convex,} \quad (9.11)$$

which ensures that our method of complex interpolation of quasi-Banach spaces applies to this setting. Now, Proposition 7.7 and the extra lattice structure on the discrete spaces allow us to use Theorem 7.9 (cf. also the remarks following its statement) and compute the intermediate interpolation spaces using Calderón's formula (9.7). Thus, for indices as in the statement of the theorem,

$$\left[f_{s_0}^{p_0,q_0}, f_{s_1}^{p_1,q_1} \right]_\theta = (f_{s_0}^{p_0,q_0})^{1-\theta}(f_{s_1}^{p_1,q_1})^\theta = f_s^{p,q}, \quad (9.12)$$

where we have also used the analogue of the last inequality in (9.7) for homogeneous spaces. Finally, thanks to Proposition 9.3, a similar argument works in the case of Besov spaces. This completes the proof of the theorem. $\square$

We now discuss the interpolation of some of the Besov and Triebel-Lizorkin spaces defined in this section and establish the following analogue of Theorems 5.1 and 9.1 in Lipschitz domains.

Theorem 9.4. *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \ne \alpha_1$, $0 < q_0, q_1, q \le \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$(F_{\alpha_0}^{p,q_0}(\Omega), F_{\alpha_1}^{p,q_1}(\Omega))_{\theta,q} = B_\alpha^{p,q}(\Omega), \quad 0 < p < \infty, \quad (9.13)$$

$$(B_{\alpha_0}^{p,q_0}(\Omega), B_{\alpha_1}^{p,q_1}(\Omega))_{\theta,q} = B_\alpha^{p,q}(\Omega), \quad 0 < p \le \infty. \quad (9.14)$$

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$ and $0 < q_0, q_1 \le \infty$ then

$$[F_{\alpha_0}^{p_0,q_0}(\Omega), F_{\alpha_1}^{p_1,q_1}(\Omega)]_\theta = F_\alpha^{p,q}(\Omega), \quad (9.15)$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
If $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \ne \alpha_1$, $0 < p_0, p_1, q_0, q_1 \le \infty$ and either $p_0 + q_0 < \infty$ or $p_1 + q_1 < \infty$ then also

$$[B_{\alpha_0}^{p_0,q_0}(\Omega), B_{\alpha_1}^{p_1,q_1}(\Omega)]_\theta = B_\alpha^{p,q}(\Omega). \quad (9.16)$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
Finally, the same interpolation results remain valid if the spaces $B_s^{p,q}(\Omega)$, $F_s^{p,q}(\Omega)$ are replaced by $B_{s,0}^{p,q}(\Omega)$ and $F_{s,0}^{p,q}(\Omega)$, respectively.

Proof. Thanks to the existence of the universal, bounded, linear extension operator reviewed in Theorem 6.3, the identities (9.13)-(9.16) follow from (9.10), Lemma 7.11 and Theorems 5.1–9.1. There remains to prove the last claim made in the statement of the theorem. To this end, we note that the operators

$$I - E_{\mathbb{R}^n \setminus \overline{\Omega}} \circ \mathcal{R}_{\mathbb{R}^n \setminus \overline{\Omega}} : F_s^{p,q}(\mathbb{R}^n) \longrightarrow F_{s,0}^{p,q}(\Omega),$$

$$I - E_{\mathbb{R}^n \setminus \overline{\Omega}} \circ \mathcal{R}_{\mathbb{R}^n \setminus \overline{\Omega}} : B_s^{p,q}(\mathbb{R}^n) \longrightarrow B_{s,0}^{p,q}(\Omega), \quad (9.17)$$

are projections onto the target spaces (i.e., are linear, bounded and idempotent). Furthermore, it is apparent that a distribution $f \in S'(\mathbb{R}^n)$ belongs to the range of $I - E_{\mathbb{R}^n \setminus \overline{\Omega}} \circ \mathcal{R}_{\mathbb{R}^n \setminus \overline{\Omega}}$ if and only if $\text{supp} f \subseteq \overline{\Omega}$. Thus, the second part of Lemma 7.11 applies and yields the desired conclusion. $\square$

Related results can be found in [**27**], [**84**].

10 Extrapolating compactness on Besov and Triebel-Lizorkin spaces

In [**17**], M. Cwikel has proved the following remarkable one-sided compactness property for the real method of interpolation for (compatible) Banach couples.

Theorem 10.1. *Let X_j, Y_j, $j = 0, 1$, be two compatible Banach couples and suppose that the linear operator $T : X_j \to Y_j$ is bounded for $j = 0$ and compact for $j = 1$. Then $T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is compact for all $\theta \in (0, 1)$ and $p \in [1, \infty]$.*

The corresponding result for the complex method of interpolation remains open. However, in [**17**] M. Cwikel has shown that the property of being compact can be extrapolated on complex interpolation scales of Banach spaces:

Theorem 10.2. *Let X_j, Y_j, $j = 0, 1$, be two compatible Banach couples and suppose that $T : X_j \to Y_j$, $j = 0, 1$, is a bounded, linear operator with the property that there exists $\theta^* \in (0, 1)$ such that $T : [X_0, X_1]_{\theta^*} \to [Y_0, Y_1]_{\theta^*}$ is compact. Then $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is compact for all values of θ in $(0, 1)$.*

It is unclear whether a similar result holds for arbitrary compatible quasi-Banach couples. We shall nonetheless show that such an extrapolation result holds for the scales of Besov and Triebel-Lizorkin spaces. More specifically, we have:

Theorem 10.3. *Let R be an open, convex subset of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ and assume that T is a linear operator such that*

$$T : B_s^{p,q}(\mathbb{R}^n) \longrightarrow B_s^{p,q}(\mathbb{R}^n), \tag{10.1}$$

is bounded whenever $(s, 1/p, 1/q) \in R$. If there exists $(s^*, 1/p^*, 1/q^*) \in R$ such that T maps $B_{s^*}^{p^*,q^*}(\mathbb{R}^n)$ compactly into itself then the operator (10.1) is in fact compact for all $(s, 1/p, 1/q) \in R$.

Moreover, a similar result holds for the scale of Triebel-Lizorkin spaces.

Proof. For each $N \in \mathbb{N}$, denote by

$$P_N : B_s^{p,q}(\mathbb{R}^n) \longrightarrow B_s^{p,q}(\mathbb{R}^n) \tag{10.2}$$

the projection onto the finite dimensional subspace generated by the first N functions in a fixed, sufficiently smooth (and suitably labeled) wavelet basis. Thus, P_N are linear operators, of finite rank which also satisfy

$$\sup_{N \in \mathbb{N}} \|P_N\|_{B_s^{p,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)} < \infty, \quad \text{and} \tag{10.3}$$

$$P_N \longrightarrow I \text{ pointwise on } B_s^{p,q}(\mathbb{R}^n) \text{ as } N \to \infty. \tag{10.4}$$

Let $\mathcal{O}$ be an arbitrary relatively compact subset of $B_s^{p,q}(\mathbb{R}^n)$. From (10.3)-(10.4) plus a standard argument, based on covering $\mathcal{O}$ with finitely many balls of sufficiently small radii, it follows that

$$P_N \longrightarrow I \text{ pointwise, as } N \to \infty, \text{ uniformly on } \mathcal{O}. \tag{10.5}$$

In particular, if

$$T : B_{s_0}^{p_0,q_0}(\mathbb{R}^n) \longrightarrow B_{s_0}^{p_0,q_0}(\mathbb{R}^n) \text{ is linear and bounded, and} \tag{10.6}$$

$$T : B_{s_1}^{p_1,q_1}(\mathbb{R}^n) \longrightarrow B_{s_1}^{p_1,q_1}(\mathbb{R}^n) \text{ is linear and compact,} \tag{10.7}$$

then there exists a finite constant $C = C(T) > 0$ and, for each $\varepsilon > 0$, an integer $N(\varepsilon) \in \mathbb{N}$ such that

$$\|T - P_N T\|_{B_{s_0}^{p_0,q_0}(\mathbb{R}^n) \to B_{s_0}^{p_0,q_0}(\mathbb{R}^n)} \leq C, \quad \forall N \in \mathbb{N}, \tag{10.8}$$

$$\|T - P_N T\|_{B_{s_1}^{p_1,q_1}(\mathbb{R}^n) \to B_{s_1}^{p_1,q_1}(\mathbb{R}^n)} \leq \varepsilon, \quad \forall N \geq N(\varepsilon). \tag{10.9}$$

Since whenever $s_0 \neq s_1$ and $\theta \in (0,1)$, Theorem 9.1 gives

$$\left[B_{s_0}^{p_0,q_0}(\mathbb{R}^n), \, B_{s_1}^{p_1,q_1}(\mathbb{R}^n) \right]_{\theta} = B_s^{p,q}(\mathbb{R}^n) \tag{10.10}$$

provided $1/p := (1-\theta)/p_0 + \theta/p_1$, $1/q := (1-\theta)/q_0 + \theta/q_1$, $s := (1-\theta)s_0 + \theta s_1$, we may conclude from this and (10.8)-(10.9) that

$$\|T - P_N T\|_{B_s^{p,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)} \leq C^{1-\theta} \varepsilon^{\theta} \tag{10.11}$$

granted that $N \geq N(\varepsilon)$. Thus, T is compact as an operator on $B_s^{p,q}(\mathbb{R}^n)$ since it can be approximated in the strong operator norm by linear operators of finite rank.

The argument in the case of Triebel-Lizorkin spaces is very similar and this finishes the proof of the theorem. $\qquad\square$

Corollary 10.4. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that R is an open, convex subset of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. Also, let T be a linear operator such that*

$$T : B_s^{p,q}(\Omega) \longrightarrow B_s^{p,q}(\Omega), \tag{10.12}$$

is bounded whenever $(s, 1/p, 1/q) \in R$. If there exists $(s^*, 1/p^*, 1/q^*) \in R$ such that T maps $B_{s^*}^{p^*,q^*}(\Omega)$ compactly into itself then the operator (10.12) is in fact compact for all $(s, 1/p, 1/q) \in R$.

Moreover, a similar result holds for the scale of Triebel-Lizorkin spaces.

Proof. Let

$$\mathcal{R}_{\Omega} : B_s^{p,q}(\mathbb{R}^n) \longrightarrow B_s^{p,q}(\Omega), \tag{10.13}$$

$$E_{\Omega} : B_s^{p,q}(\Omega) \longrightarrow B_s^{p,q}(\mathbb{R}^n), \tag{10.14}$$

be, respectively, the operator of restriction to Ω, and Rychkov's universal extension operator. Then

$$\widetilde{T} := E_{\Omega} \circ T \circ \mathcal{R}_{\Omega} : B_s^{p,q}(\mathbb{R}^n) \longrightarrow B_s^{p,q}(\mathbb{R}^n) \tag{10.15}$$

is well-defined and bounded for whenever $(s, 1/p, 1/q) \in R$. From assumptions, it also maps $B_{s^*}^{p^*,q^*}(\mathbb{R}^n)$ compactly into itself. Hence, by Theorem 10.3, the operator (10.15) is compact for each $(s, 1/p, 1/q) \in R$. Since $T = \mathcal{R}_{\Omega} \circ \widetilde{T} \circ E_{\Omega}$, the desired conclusion follows. $\qquad\square$

Next, recall that $(a)_+ := \max\{a, 0\}$ and consider three parameters p, q, s subject to

$$0 < p, q \leq \infty, \qquad (n-1)\left(\tfrac{1}{p} - 1\right)_+ < s < 1. \tag{10.16}$$

Then, given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, one can define the Besov space $B_s^{p,q}(\partial\Omega)$ by transporting its Euclidean counterpart to $\partial\Omega$ via localization and pull-back. The relation between this space and the Besov and Triebel-Lizorkin scales in Ω is made apparent in the following theorem, proved in [56], [57].

Theorem 10.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p, s satisfy $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then the following hold:*

(i) The restriction to the boundary extends to a linear, bounded operator

$$\text{Tr} : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_s(\partial\Omega) \quad \text{for} \quad 0 < q \leq \infty. \tag{10.17}$$

Moreover, for this range of indices, Tr is onto and has a bounded right inverse

$$\text{Ex} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{10.18}$$

(ii) Similar considerations hold for

$$\text{Tr} : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_s(\partial\Omega) \tag{10.19}$$

(it is understood that $q = \infty$ if $p = \infty$). Furthermore, if $\min\{p,1\} \leq q \leq \infty$, then the operator (10.19) has a linear, bounded right inverse

$$\text{Ex} : B^{p,p}_s(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{10.20}$$

Let $\mathcal{O}$ be the collection of all triplets (p, q, s) such that (10.16) holds. A particular consequence of Theorem 10.5 is that the scale $B^{p,q}_s(\partial\Omega)$ is a retract of $B^{p,q}_{s+1/p}(\Omega)$, when both are indexed by $(p, q, s) \in \mathcal{O}$. Then, from Corollary 10.4 and the same type of argument as in its proof, we obtain the following.

Theorem 10.6. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that R is an open, convex subset of $\mathcal{O}$ (defined above). Also, let T be a linear operator such that*

$$T : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_s(\partial\Omega), \tag{10.21}$$

is bounded whenever $(s, 1/p, 1/q) \in R$. If there exists $(s^*, 1/p^*, 1/q^*) \in R$ such that T maps $B^{p^*,q^*}_{s^*}(\partial\Omega)$ compactly into itself then the operator (10.21) is in fact compact for all $(s, 1/p, 1/q) \in R$.

We now wish to elaborate on the significance of this result in the context of PDE's. For the remainder of this section assume that Ω is a bounded domain in $\mathbb{R}^n$ and that $\mathcal{K}$ is as in (8.13). In their celebrated 1978 Acta paper [**30**], Fabes, Jodeit, and Rivière have shown that

$$\partial\Omega \in C^1 \Longrightarrow \mathcal{K} \text{ is compact on } L^p(\partial\Omega) \text{ for each } p \in (1, \infty). \tag{10.22}$$

Since $\mathcal{K}$ also turns out to be well-defined and bounded on the Sobolev space $W^{1,p}(\partial\Omega)$, $1 < p < \infty$, it follows from this and real interpolation that

$$\partial\Omega \in C^1 \Longrightarrow \mathcal{K} \text{ is compact on } B^{p,q}_s(\partial\Omega) \text{ for each } p, q \in (1, \infty) \text{ and } s \in (0, 1). \tag{10.23}$$

Furthermore, it has been established in [**57**] that, for each Lipschitz domain Ω,

$$\mathcal{K} \text{ is bounded on } B^{p,q}_s(\partial\Omega) \text{ whenever } p, q, s \text{ are as in (10.16).} \tag{10.24}$$

In concert, (10.24), (10.23) and Theorem 10.6 yield the following result.

Theorem 10.7. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ with a C^1 boundary. Then*

$$\mathcal{K} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_s(\partial\Omega) \tag{10.25}$$

is a compact operator whenever p, q, s are as in (10.16).

The importance of this theorem is most apparent in the context of the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \ \Omega, \quad u \in B_{s+1/p}^{p,q}(\Omega), \quad \text{Tr}\, u = f \in B_s^{p,q}(\partial\Omega), \tag{10.26}$$

in a bounded, C^1 domain $\Omega \subset \mathbb{R}^n$. Concretely, if for the problem (10.26) one once again seeks solution expressed as in (8.15), then a crucial ingredient in this approach is establishing that

$$\tfrac{1}{2}I + \mathcal{K} : B_s^{p,q}(\partial\Omega) \longrightarrow B_s^{p,q}(\partial\Omega) \tag{10.27}$$

is invertible whenever p, q, s are as in (10.16). In this regard, Theorem 10.7 gives that the operator (10.27) is Fredholm with index zero which is the hardest step in the proof of the fact that (10.27) is invertible.

While we shall not fully pursue this issue here, we would like to point out that the well-posedness of (10.26) for $\Omega \subset \mathbb{R}^n$ bounded C^1 domain, and p, q, s as in (10.16), is a key ingredient in the proof of the last claim in the statement of Theorem 1.1.

11 Estimates for subaveraging functions

Recall that, for an open set $\Omega \subset \mathbb{R}^n$, we have introduced $\delta(x) := \text{dist}\,(x, \partial\Omega)$. In order to facilitate the subsequent discussion, we make the following

Definition 11.1. *Assume that Ω is an open subset of $\mathbb{R}^n$ and that $0 < p < \infty$. A function $u \in L_{\text{loc}}^p(\Omega)$ is p-subaveraging if there exists a positive constant C with the following property:*

$$|u(x)| \leq C \left(\frac{1}{|B_r|} \int_{B_r(x)} |u(y)|^p \, dy \right)^{\frac{1}{p}} \tag{11.1}$$

for almost every $x \in \Omega$ and all $r \in (0, \delta(x))$.

Employing ideas first developed by Fefferman and Stein in [**32**], the following result can be proved.

Lemma 11.2. *If there exists $p_0 > 0$ such that u is p_0-subaveraging function, then u is p-subaveraging for every $p \in (0, \infty)$.*

Granted this, it is unequivocal to refer to a function u as simply being *sub-averaging* if it is p-subaveraging for some $p \in (0, \infty)$. The optimal constants which can be used in (11.1) make up what we call the subaveraging character of the function u.

There are clear connections between the subaveraging property and reverse Hölder estimates. To illustrate this, we state the following.

Lemma 11.3. *Let u be a subaveraging function in a domain $\Omega \subset \mathbb{R}^n$ and assume that $0 < p, q < \infty$. Then*

$$\left(\frac{1}{|B_r|} \int_{B_r(x)} |u(y)|^q \, dy \right)^{\frac{1}{q}} \leq C \left(\frac{1}{|B_{2r}|} \int_{B_{2r}(x)} |u(y)|^p \, dy \right)^{\frac{1}{p}} \tag{11.2}$$

uniformly for $x \in \Omega$ and $0 < r < \delta(x)/2$, where the constant C depends only on p, q, n and the subaveraging character of u.

Proof. We write

$$\frac{1}{|B_r|}\int_{B_r(x)}|u(y)|^q\,dy \;\le\; C\frac{1}{|B_r|}\int_{B_r(x)}\left(\frac{1}{|B_r|}\int_{B_r(y)}|u(z)|^p\,dz\right)^{q/p}dy$$

$$\le\; C\frac{1}{|B_r|}\int_{B_r(x)}\left(\frac{1}{|B_r|}\int_{B_{2r}(x)}|u(z)|^p\,dz\right)^{q/p}dy$$

$$\le\; C\left(\frac{1}{|B_{2r}|}\int_{B_{2r}(x)}|u(z)|^p\,dz\right)^{q/p} \tag{11.3}$$

from which (11.2) readily follows. $\qquad\square$

Assume next that L is as in (4.11)-(4.12) and, for a given open set $\Omega\subset\mathbb{R}^n$, denote by $\mathrm{Ker}\,L$ the space of all C^∞ functions satisfying $Lu=0$ in Ω. We present some useful interior estimates, which are essentially folklore.

Lemma 11.4. *Let L be an elliptic differential operator as above and assume that $\Omega\subset\mathbb{R}^n$ is open. Then for each $u\in\mathrm{Ker}\,L$, $0<p<\infty$, $k\in\mathbb{N}_o$, and $x\in\Omega$, $0<r<\delta(x)$,*

$$|\nabla^k u(x)|^p \le \frac{C}{r^{n+kp}}\int_{B_r(x)}|u(y)|^p\,dy \tag{11.4}$$

where $C=C(L,p,k,n)>0$ is a finite constant. In particular,

$$u\in\mathrm{Ker}\,L \implies u \text{ is subaveraging.} \tag{11.5}$$

Proof. Rescaling (11.4), there is no loss of generality in assuming that $r=1$. In this scenario, (11.4) with $p=2$ follows from Sobolev's embedding theorem and standard interior regularity results (cf., e.g., Theorem 11.1, p. 379 in [**80**]). In particular, if $u\in\mathrm{Ker}\,L$ then u is 2-subaveraging and, hence, (11.5) holds, thanks to Lemma 11.2. With this in hand we then write

$$r^k|\nabla^k u(x)| \le C\left(\frac{1}{|B_r|}\int_{B_{r/2}(x)}|u(y)|^2\,dy\right)^{1/2} \le C\left(\frac{1}{|B_r|}\int_{B_r(x)}|u(y)|^p\,dy\right)^{1/p} \tag{11.6}$$

by Lemma 11.3. $\qquad\square$

Lemma 11.5. *Let $\Omega\subset\mathbb{R}^n$ be an arbitrary bounded, star-like Lipschitz domain and consider $0<q\le p<\infty$, $sp>-1$. Then for a subaveraging function u in Ω,*

$$\left(\int_\Omega |u_{\mathrm{rad}}(x)|^p\,\delta(x)^{sp}\,dx\right)^{1/p} \le \kappa\left(\int_\Omega |u(x)|^q\,\delta(x)^{sq+n(\frac{q}{p}-1)}\,dx\right)^{1/q} \tag{11.7}$$

where κ depends only on p,q,s,n, the Lipschitz character of Ω and the subaveraging character of u.

Proof. The argument below follows closely [**79**]. Nonetheless, some alterations allow to treat a wider range of indices so we include a complete argument. We begin by considering the Whitney decomposition of the domain Ω (cf. [**78**]). Specifically, there exists a sequence of balls $\{B_{r_j}\}_j$ with the following properties:

$$\delta(x)\approx r_j \text{ for } x\in B_{r_j}, \qquad \Omega=\cup_j B_{r_j}, \qquad \sum_j \chi_j \le 200^{2n}, \tag{11.8}$$

where χ_j denotes the characteristic function of the ball B_{r_j}. Then Fatou's lemma implies

$$\int_\Omega \delta(x)^{sp}(|u|^p)_{\mathrm{rad}}(x)\,dx \leq C \sum_j \int_\Omega \delta(x)^{sp}(\chi_j|u|^p)_{\mathrm{rad}}(x)\,dx, \tag{11.9}$$

Next set

$$\widehat{B}_{r_j} := \{z \in \Omega : e^{-t}z \in B_{r_j} \text{ for some } t > 0\}, \tag{11.10}$$

and note that $\mathrm{supp}\,(\chi_j|u|^p)_{\mathrm{rad}} \subseteq \widehat{B}_{r_j}$ and $\|(\chi_j|u|^p)_{\mathrm{rad}}\|_{L^\infty(\widehat{B}_{r_j})} \leq \|u\|^p_{L^\infty(B_{r_j})}$. Also,

$$\int_{\widehat{B}_{r_j}} \delta(x)^{sp}\,dx \leq C\,r_j^{n-1} \int_0^{cr_j} t^{sp}\,dt = C r_j^{n+sp}. \tag{11.11}$$

Consequently, granted the sub-averaging property of u,

$$\int_\Omega \delta(x)^{sp}(\chi_j|u|^p)_{\mathrm{rad}}(x)\,dx \quad\leq\quad Cr_j^{n+sp} \sup_{x \in B_{r_j}} |u(x)|^p \leq C r_j^{n+sp}\Big(\frac{1}{r_j^n}\int_{B_{2r_j}}|u(x)|^q\,dx\Big)^{p/q}$$

$$\leq\quad C\Big(\int_{B_{2r_j}}(\delta(x)^{s+n(1/p-1/q)}|u(x)|)^q\,dx\Big)^{p/q}, \tag{11.12}$$

where the last inequality holds thanks to (11.8). Recalling that no point of Ω is contained in more than 200^{2n} of the balls B_{2r_j}, we can combine (11.9) and (11.12). The assumption $p \geq q$ allows to sum up the integrals in (11.12), which leads to (11.7). $\qquad\square$

Lemma 11.6. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$ and $0 < q \leq p < \infty$, $s \in \mathbb{R}$. Then for every subaveraging function u in Ω,*

$$\Big(\int_\Omega [\delta(x)^{s+n(\frac{1}{q}-\frac{1}{p})}|u(x)|]^p\,dx\Big)^{1/p} \leq \kappa\Big(\int_\Omega [\delta(x)^s|u(x)|]^q\,dx\Big)^{\frac{1}{q}} \tag{11.13}$$

where κ depends exclusively on p, q, s, n, and the subaveraging character of u.

Proof. To start, let us observe that for every $x \in \Omega$

$$|u(x)|^q \leq C\delta(x)^{-n-qs}\int_{B_{\delta(x)/2}(x)}(\delta(y)^s|u(y)|)^q\,dy \leq C\delta(x)^{-n-qs}\int_\Omega(\delta(y)^s|u(y)|)^q\,dy, \tag{11.14}$$

owing to the sub-averaging property of u and a simple observation, to the effect that $\delta(x) \approx \delta(y)$ for every $y \in B_{\delta(x)/2}$. Next,

$$|u(x)| = |u(x)|^\theta|u(x)|^{1-\theta} \leq C|u(x)|^\theta\delta(x)^{-(\frac{n}{q}+s)(1-\theta)}\Big(\int_\Omega(\delta(y)^s|u(y)|)^q\,dy\Big)^{\frac{1-\theta}{q}}, \tag{11.15}$$

for every θ such that $0 < \theta \leq 1$. Consequently,

$$\delta(x)^{p\left[\sigma+n\left(\frac{1}{q}-\frac{1}{p}\right)\right]}|u(x)|^p \leq C|u(x)|^{\theta p}\,\delta(x)^{-n+\theta p(\sigma+\frac{n}{q})}\Big(\int_\Omega(\delta(y)^s|u(y)|)^q\,dy\Big)^{(1-\theta)\frac{p}{q}}. \tag{11.16}$$

At this stage, we set $\theta := \frac{q}{p}$ (so that $0 < \theta \leq 1$ given that $q \leq p$) and integrate both sides of the above inequality in order to obtain (11.13). $\qquad\square$

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Lemma 11.7. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that L is an elliptic operator as in (4.11). Also, fix $0 < q \leq p < \infty$, $s \in \mathbb{R}$, and $k \in \mathbb{N}_o$. Then for any $u \in \operatorname{Ker} L$,*

$$\left(\int_\Omega (\delta(x)^{s+k+n(\frac{1}{p}-\frac{1}{q})} |\nabla^k u(x)|)^p \, dx \right)^{1/p} \leq C \left(\int_\Omega (\delta(x)^s |u(x)|)^q \, dx \right)^{1/q}, \tag{11.17}$$

where $C = C(L, \Omega, p, q, s, k) > 0$ is a finite constant. In particular, when $q = p$,

$$\left(\int_\Omega (\delta(x)^{s+k} |\nabla^k u(x)|)^p \, dx \right)^{1/p} \leq C \left(\int_\Omega (\delta(x)^s |u(x)|)^p \, dx \right)^{1/p} \tag{11.18}$$

holds for any $u \in \operatorname{Ker} L$ provided $0 < p < \infty$, $s \in \mathbb{R}$, and $k \in \mathbb{N}_o$.

Proof. Start with a Whitney decomposition of Ω into a family of balls as in (11.8). Integrating (11.4) allows us to write

$$\int_{B_{r_j}} (r_j^{s+k} |\nabla^k u(x)|)^p \, dx \leq C \int_{B_{2r_j}} (r_j^s |u(x)|)^p \, dx \tag{11.19}$$

which further translates into

$$\int_{B_{r_j}} (\delta(x)^{s+k} |\nabla^k u(x)|)^p \, dx \leq C \int_{B_{2r_j}} (\delta(x)^s |u(x)|)^p \, dx \tag{11.20}$$

for each j. This, in turn, readily yields (11.18) by summing over all j's. Having proved (11.18), the estimate (11.17) follows by bringing in (11.13) and (11.5) and by re-denoting s by $s + n(\frac{1}{q} - \frac{1}{p})$. $\qquad\square$

12 Spaces of null-solutions of elliptic operators

We start by presenting the

Proof of Theorem 1.4. The identity (1.14) is proved in [**79**] for $n = 2$ and $L = \overline{\partial}$, the Cauchy-Riemann operator in the plane, and the proof goes through virtually verbatim. In fact, (1.15) is also proved in [**79**] (once again, when $L = \overline{\partial}$), but for a different concept of complex interpolation. For the reasoning in [**79**] to work in our case (i.e., for the complex method of interpolation described in §7), we need to check two more things, namely:

(i) given $\alpha_0, \alpha_1 \in \mathbb{R}$ and $0 < p < \infty$, the space $\mathbb{H}^p_{\alpha_0}(\Omega; L) + \mathbb{H}^p_{\alpha_1}(\Omega; L)$ is analytically convex, and

(ii) given $f \in L^p(\Omega) \cap \operatorname{Ker} L$, $0 < p < \infty$, the $L^p(\Omega) \cap \operatorname{Ker} L$-valued mapping

$$\{\zeta \in \mathbb{C} : \Re e \, \zeta > 0\} \ni \zeta \mapsto R_\zeta f \in L^p(\Omega) \cap \operatorname{Ker} L, \quad \text{where} \tag{12.1}$$

$$(R_\zeta f)(x) := \frac{(-1)^m}{\Gamma(\zeta + m)} \int_0^\infty t^{\zeta+m-1} \frac{d^m}{dt^m} \left(e^{-t} f(e^{-t} x) \right) dt, \quad x \in \Omega, \tag{12.2}$$

is analytic, in the sense of (7.3). Here, Ω is assumed to be star-like with respect to the origin in $\mathbb{R}^n$, $\Gamma(\cdot)$ is the Gamma function and $m \in \mathbb{N}_o$. Note that the definition of the fractional integration operator R_ζ is, in fact, independent of m, as an m-fold integration by parts shows.

Condition (i) is required since, according to the setup discussed in §7, the sum of the two quasi-Banach spaces in question must be analytically convex. The fact that the map (12.1) has a complex derivative at every point in the right half-plane is proved in Lemma 9 on p. 663 in [**79**]. Nonetheless, as pointed out right after (7.3), this concept of analyticity is too weak for our method of complex interpolation to

work. Thus, instead of pointwise complex differentiability, we have to insist on a power series condition such as (7.3), hence the necessity of (ii). Granted (i)-(ii), the same argument as in [**79**] applies and yields (1.15) (strictly speaking, this result is proved for analytic functions in [**79**] but the argument goes through with only minor alterations in the case of null-solutions for the operators L).

We now turn to the proof of (i). Since $\mathbb{H}^p_{\alpha_0}(\Omega; L) + \mathbb{H}^p_{\alpha_1}(\Omega; L) = \mathbb{H}^p_{\alpha_0}(\Omega; L)$ if $-\infty < \alpha_0 < \alpha_1 < +\infty$, this claim reduces to showing that

$$\mathbb{H}^p_\alpha(\Omega; L) \text{ is analytically convex, } \forall\, \alpha \in \mathbb{R}, \ \forall\, p \in (0, \infty). \tag{12.3}$$

To justify (12.3), first introduce

$$L^p(\Omega, \delta^s) := \left\{ u \in L^p_{\text{loc}}(\Omega) : \int_\Omega |u(x)|^p \operatorname{dist}(x, \partial\Omega)^{sp}\, dx < +\infty \right\}, \tag{12.4}$$

equipped with the obvious quasi-norm. Next, we note that the map

$$\mathbb{H}^p_\alpha(\Omega; L) \ni u \mapsto (\nabla^{\langle\alpha\rangle} u, \nabla^{\langle\alpha\rangle-1} u, \ldots u) \in L^p(\Omega, \delta^{\langle\alpha\rangle-\alpha}) \oplus L^p(\Omega) \oplus \cdots \oplus L^p(\Omega) \tag{12.5}$$

identifies $\mathbb{H}^p_\alpha(\Omega; L)$ with a closed subspace of $L^p(\Omega, \delta^{\langle\alpha\rangle-\alpha}) \oplus L^p(\Omega) \oplus \cdots \oplus L^p(\Omega)$. Since, by Lemma 7.6, the latter space is analytically convex, it follows that $\mathbb{H}^p_\alpha(\Omega; L)$ is analytically convex as well.

Next, consider the analyticity claim made in (ii) about the operator (12.1)-(12.2). We shall choose $m = 1$ and drop the factor in front of the integral so that, for a fixed $f \in L^p(\Omega) \cap \operatorname{Ker} L$, it suffices to consider

$$\{\zeta \in \mathbb{C} : \Re e\, \zeta > 0\} \ni \zeta \mapsto \int_0^\infty t^\zeta e^{-t}\Big(1 + x \cdot (\nabla f)(e^{-t}x)\Big)\, dt, \quad x \in \Omega. \tag{12.6}$$

Continuing our series of reductions, it will suffice to assume that $0 < p < 1$ and justify that, for a fixed $f \in L^p(\Omega) \cap \operatorname{Ker} L$,

$$\{\zeta \in \mathbb{C} : \Re e\, \zeta > 0\} \ni \zeta \mapsto A_\zeta f(\cdot) := \int_0^\infty t^\zeta e^{-t} (\nabla f)(e^{-t}\cdot)\, dt \tag{12.7}$$

is a $L^p(\Omega)$-valued analytic function (the lower order term in (12.6) is handled similarly).

To this end, we fix $\zeta_o \in \mathbb{C}$ with $\Re e\, \zeta_o > 0$ and formally write

$$A_\zeta f = \sum_{j=0}^\infty (\zeta - \zeta_o)^j f_j \tag{12.8}$$

where

$$f_j(x) := \frac{1}{j!} \int_0^\infty (\ln t)^j\, t^{\zeta_o}\, e^{-t} (\nabla f)(e^{-t}x)\, dt, \qquad x \in \Omega, \ j \in \mathbb{N}_o. \tag{12.9}$$

There remains to show that $f_j \in L^p(\Omega)$ and that the series (12.8) converges uniformly for $|\zeta - \zeta_o| < \eta$, provided that $\eta > 0$ is small enough. This, in turn, will follow as soon as we show that there exists a finite constant $\kappa = \kappa(\zeta_o, p, f) > 0$ and some $C > 0$ such that

$$\|f_j\|_{L^p(\Omega)} \le C\kappa^j, \qquad \forall\, j \in \mathbb{N}_o. \tag{12.10}$$

To see this, we first make the elementary observation that, for each $\varepsilon > 0$, there exists a finite constant $C_\varepsilon > 0$ such that for each $j \in \mathbb{N}_o$,

$$\frac{t^\varepsilon |\ln t|^j}{j!} \le (C_\varepsilon)^j \ \text{ if } 0 < t \le 1, \quad \text{and} \quad \frac{|\ln t|^j e^{-\varepsilon t}}{j!} \le (C_\varepsilon)^j \ \text{ if } 1 < t < \infty. \tag{12.11}$$

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Indeed, in the second inequality in (12.11), use $|\ln t|^j \leq t^j$ for $t \in (1, \infty)$ and the fact that the mapping $(0, \infty) \ni t \mapsto t^j e^{-\varepsilon t} \in (0, \infty)$ has a global maximum at $t = j/\varepsilon$, to write for each $t \in (1, \infty)$

$$\frac{|\ln t|^j e^{-\varepsilon t}}{j!} \leq \varepsilon^{-j} \frac{j^j e^{-j}}{j!} \leq C\Big(\frac{1}{\varepsilon}\Big)^j, \qquad j \in \mathbb{N}, \tag{12.12}$$

where the last step uses Stirling's formula to the effect that $j! \approx j^j e^{-j}$. Finally, the first inequality in (12.11) reduces to what we have just proved by making the substitution $t = e^{-\tau}$, $\tau \in (0, \infty)$.

Turning to the proof of (12.10) in the earnest, we note that, by assumption, there exists a Lipschitz function $\varphi : S^{n-1} \to \mathbb{R}$ with $\inf \{\varphi(\omega) : \omega \in S^{n-1}\} > 0$ such that $\Omega = \{\omega\rho : \omega \in S^{n-1}, 0 < \rho < \varphi(\omega)\}$. As a consequence of the Mean Value Theorem for null-solutions of L, there exists a small, compact neighborhood $\mathcal{O}$ of $0 \in \Omega$ such that, whenever $0 < p < \infty$,

$$\int_\Omega |u(x)|^p \, dx \leq C \int_{\Omega \setminus \mathcal{O}} |u(x)|^p \, dx, \quad \text{uniformly for } u \in \operatorname{Ker} L. \tag{12.13}$$

Note that $Lf_j = 0$ in Ω for every $j \in \mathbb{N}_o$ so that, for some $M > 0$ large enough,

$$
\begin{aligned}
\int_\Omega |f_j(x)|^p \, dx \;\leq\;& C \int_{\Omega \setminus \mathcal{O}} |f_j(x)|^p \, dx \\[4pt]
\leq\;& C \int_{S^{n-1}} \int_{e^{-M}\varphi(\omega)}^{\varphi(\omega)} |f_j(\rho\omega)|^p \rho^{n-1} \, d\rho d\omega \\[4pt]
\leq\;& (C_\varepsilon)^j \int_{S^{n-1}} \int_{e^{-M}\varphi(\omega)}^{\varphi(\omega)} \Big(\int_0^\infty t^{\Re e\, \zeta_o - \varepsilon} e^{-(1-\varepsilon)t} |(\nabla f)(e^{-t}\rho\omega)| \, dt\Big)^p \rho^{n-1} \, d\rho d\omega \\[4pt]
\leq\;& (C_\varepsilon)^j \int_{S^{n-1}} \int_{e^{-M}\varphi(\omega)}^{\varphi(\omega)} \Big(\int_0^\infty t^{\Re e\, \zeta_o - \varepsilon} e^{-(1-\varepsilon)t} |(\nabla f)_{\mathrm{rad}}(e^{-t}\rho\omega)| \, dt\Big)^p \, d\rho d\omega.
\end{aligned}
\tag{12.14}
$$

Above, we picked $0 < \varepsilon < \min\{1, \Re e\, \zeta_o\}$ and then made use of the estimate (12.11). Also, $(\cdot)_{\mathrm{rad}}$ is the maximal radial operator introduced in (6.4).

To proceed, change variables such that $\rho = \varphi(\omega) e^{-s}$, $s \in (0, M)$, and $\lambda = s + t$ so that $\lambda \in (s, \infty)$; in particular, $\lambda > t > \lambda - M$. Then the last expression in (12.14) is

$$
\begin{aligned}
\leq\;& (C_\varepsilon)^j \int_{S^{n-1}} \int_0^M \Big(\int_s^\infty \lambda^{\Re e\, \zeta_o - \varepsilon} e^{-(1-\varepsilon)t} |(\nabla f)_{\mathrm{rad}}(e^{-\lambda}\varphi(\omega)\omega)| \, d\lambda\Big)^p \, ds\, d\omega \\[4pt]
\leq\;& (C_\varepsilon)^j \int_{S^{n-1}} \int_0^\infty \Big(\lambda^{\Re e\, \zeta_o - \varepsilon + 1} e^{-(1-\varepsilon)t} |(\nabla f)_{\mathrm{rad}}(e^{-\lambda}\varphi(\omega)\omega)|\Big)^p \, d\lambda\, d\omega \\[4pt]
\leq\;& (C_\varepsilon)^j \left\{\int_{S^{n-1}} \int_0^M \Big(\lambda^{\Re e\, \zeta_o - \varepsilon + 1} e^{-(1-\varepsilon)t} |(\nabla f)_{\mathrm{rad}}(e^{-\lambda}\varphi(\omega)\omega)|\Big)^p \, d\lambda\, d\omega + \|f\|_{L^p(\Omega)}^p\right\}.
\end{aligned}
$$

The second estimate above is based on Hardy's inequality (in the version discussed in Lemma 1 on p. 670 of [**79**]). Also, in the last inequality above, we have used the fact that $(\nabla f)_{\mathrm{rad}}(e^{-\lambda}\varphi(\omega)\omega) \leq C\|f\|_{L^p(\Omega)}$, uniformly for $\lambda > M$ and $\omega \in S^{n-1}$, given that $Lf = 0$ in Ω and that $e^{-\lambda}\varphi(\omega)\omega$ lies, for this range of parameters, in a fixed, compact subset of Ω.

There remains to handle the double integral inside the brackets. For this we make the change of variables $r = e^{-\lambda}\varphi(\omega)$ so that $d\lambda \approx r^{n-1} dr$ and note that

$$\delta(r\omega) = \operatorname{dist}(r\omega, \partial\Omega) = \varphi(\omega) - r \approx e^{-\lambda} - 1 \approx \lambda, \tag{12.15}$$

uniformly for $\lambda \in (0, M)$. Thus, the double integral in question is majorized by

$$C \int_\Omega [\delta |(\nabla f)_{\mathrm{rad}}|]^p \, dx \leq C \int_\Omega [\delta |\nabla f|]^p \, dx \leq C \int_\Omega |f|^p \, dx, \tag{12.16}$$

by Lemma 11.5 and Lemma 11.6. This completes the justification of (12.10) and finishes the proof of Theorem 1.4. $\qquad\square$

Next, we turn our attention to the

Proof of Theorem 1.5. Set

$$X_i := F_{\alpha_i}^{p_i, q_i}(\Omega), \quad Z_i := F_{\alpha_i - m, 0}^{p_i, q_i}(Q), \quad Y_i = F_{\alpha_i - m, 0}^{p_i, q_i}(Q \setminus \overline{\Omega}), \tag{12.17}$$

where Q stands for some open cube in $\mathbb{R}^n$ containing $\overline{\Omega}$ and $i = 1, 2$. The intention is to use Theorem 7.10 so the first order of business is to check that the spaces $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex. First, the sum of spaces $X_0 + X_1$ is analytically convex by the argument in the proof of Theorem 9.4. As for the sum $Y_0 + Y_1$, we define the operators

$$P_1 := I - E_{\mathbb{R}^n \setminus \overline{Q}} \circ \mathcal{R}_{\mathbb{R}^n \setminus \overline{Q}} : F_{\alpha_i - m}^{p_i, q_i}(\mathbb{R}^n) \longrightarrow F_{\alpha_i - m, 0}^{p_i, q_i}(Q),$$

$$P_2 := I - E_\Omega^Q \circ \mathcal{R}_\Omega : F_{\alpha_i - m, 0}^{p_i, q_i}(Q) \longrightarrow F_{\alpha_i - m, 0}^{p_i, q_i}(Q \setminus \overline{\Omega}), \tag{12.18}$$

where $i = 0, 1$ and E_Ω^Q denotes the Rychkov's extension operator truncated near $\overline{\Omega}$ so that it maps the functions from $F_{\alpha_i - m}^{p_i, q_i}(\Omega)$ to those supported in the cube Q, i.e. belonging to Z_i. Both P_1 and P_2 are common projections for the corresponding spaces and hence Lemma 7.11 applies and shows that $Y_0 + Y_1$ are analytically convex.

Consider the operator $D := L \circ E_\Omega^Q$. One can observe now that, in the notation of Theorem 7.10, $X_i(D) = \mathrm{Ker}\, L \cap F_{\alpha_i}^{p_i, q_i}(\Omega)$ for $i = 1, 2$. We now let G stand for the composition $\mathcal{R}_\Omega \circ \Pi_L$ where Π_L is the Newtonian potential operator associated with L (defined as in (4.6) for $a(\xi) := L(-i\xi)^{-1}$) and, as usual, $\mathcal{R}_\Omega$ is the operator of restriction to Ω). Then $G : Z_i \to X_i$ boundedly, thanks to Corollary 4.3. Furthermore, if I stands for the identity operator, then for every test function $\psi \in C_c^\infty(\Omega)$ and for every distribution $u \in Z_i$,

$$\begin{aligned}\langle (D \circ G - I)u, \psi \rangle &= \langle L \circ E_\Omega^Q \circ \mathcal{R}_\Omega \circ \Pi_L u, \psi \rangle - \langle u, \psi \rangle \\ &= (-1)^\mu \langle \Pi_L u, L\psi \rangle - \langle u, \psi \rangle = 0.\end{aligned} \tag{12.19}$$

Hence, $K := D \circ G - I$ is a bounded linear operator from Z_i to Y_i and $D \circ G = I + K$ on Z_i by construction. Then (1.16) and (1.18) follow from Theorem 7.10. A similar argument works for the harmonic Besov scale and this finishes the proof of the theorem. $\qquad\square$

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., 12 (1959), 623–727.

[2] E. Albrecht, V. Müller, *Spectrum of interpolated operators*, Proc. Amer. Math. Soc., 129 (2001), no. 3, 807–814.

[3] A. B. Aleksandrov, *Essays on nonlocally convex Hardy classes*, Complex analysis and spectral theory (Leningrad, 1979/1980), pp. 1–89, Lecture Notes in Math., No. 864, Springer, Berlin-New York, 1981.

[4] J. Bergh and J. Löfström, *Interpolation Spaces. An introduction*, Springer-Verlag, 1976.

[5] A. Bernal and J. Cerdà, *Complex interpolation of quasi-Banach spaces with an A-convex containing space,* Ark. Mat. 29 (1991), 183–201

[6] R. Brown, *The Neumann problem on Lipschitz domains in Hardy spaces of order less than one*, Pacific J. Math., 171 (1995), no. 2, 389–407.

[7] A. Calderón, *Boundary value problems for the Laplace equation in Lipschitzian domains*, Recent Progress in Fourier Analysis (El Escorial, 1983), pp. 33–48, North-Holland Math. Stud., No. 111, North-Holland, Amsterdam, 1985.

[8] A. Calderón, *Lebesgue spaces of differentiable functions and distributions*, pp. 33–49 in Proc. Sympos. Pure Math., Vol. IV, 1961.

[9] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math., 24 (1964), 113–190.

[10] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Adv. in Math., 24 (1977), 101–171.

[11] W. Cao and Y. Sagher, *Stability of Fredholm properties on interpolation scales*, Ark. för Math., 28 (1990), 249–258.

[12] D.-C. Chang, G. Dafni and E.M. Stein, *Hardy spaces,* BMO*, and boundary value problems for the Laplacian on a smooth domain in $\mathbb{R}^n$*, Trans. Amer. Math. Soc., 351 (1999), 1605–1661.

[13] D.-C. Chang, S.G. Krantz and E.M. Stein, *H^p theory on a smooth domain in $\mathbb{R}^N$ and elliptic boundary value problems*, J. Funct. Anal., 114 (1993), no. 2, 286–347.

[14] D.-C. Chang, S.G. Krantz and E.M. Stein, *Hardy spaces and elliptic boundary value problems*, The Madison Symposium on Complex Analysis, pp. 119–131, Contemp. Math., Vol. 137, AMS, Providence, RI, 1992.

[15] R. Coifman, A. McIntosh and Y. Meyer, *L'intégrale de Cauchy definit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. Math., 116 (1982), 361–388.

[16] M. Cwikel and P. Nilsson, *Interpolation of weighted Banach lattices*, Technion Preprint Series No. MT-834.

[17] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J., 65 (1992), no. 2, 333–343.

[18] M. Cwikel, M. Milman and Y. Sagher, *Complex interpolation of some quasi-Banach spaces*, J. Funct. Anal., 65 (1986), 339–347.

[19] B. Dahlberg, *L^q-estimates for Green potentials in Lipschitz domains*, Math. Scand., 44 (1979), no. 1, 149–170.

[20] B. Dahlberg and C. Kenig, *Hardy spaces and the L^p–Neumann problem for Laplace's equation in a Lipschitz domain*, Ann. of Math., 125 (1987), 437–465.

[21] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 61, SIAM, Philadelphia, PA, 1992.

[22] W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann, *The complex convexity of quasi-normed linear spaces*, J. Funct. Anal., 55 (1984), 110–150.

[23] S. Dahlke and R. A. DeVore, *Besov regularity for elliptic boundary value problems*, Comm. Partial Differential Equations, 22 (1997), 1–16.

[24] R.A. DeVore, *Nonlinear approximation*, pp. 51–150 in Acta Numer., No. 7, Cambridge Univ. Press, Cambridge, 1998.

[25] R.A. DeVore, B. Jawerth and V. Popov, *Compression of wavelet decompositions*, Amer. J. Math., 114 (1992), 737–785.

[26] R.A. DeVore, G.C. Kyriazis and P. Wang, *Multiscale characterizations of Besov spaces on bounded domains*, J. Approx. Theory, 93 (1998), 273–292.

[27] R.A. DeVore and V.A. Popov, *Interpolation of Besov spaces*, Trans. Amer. Math. Soc., 305 (1988), no. 1, 397–414.

[28] R.A. DeVore, R.C. Sharpley, *Maximal Functions Measuring Smoothness*, Mem. Amer. Math. Soc., Vol. 47, No. 293, 1984.

[29] R.A. DeVore, R.C. Sharpley, *Besov spaces on domains in $\mathbb{R}^d$*, Trans. Amer. Math. Soc., 335 (1993), no. 2, 843–864.

[30] E.B. Fabes, M. Jodeit Jr. and N.M. Rivière, *Potential techniques for boundary value problems on C^1-domains*, Acta Math., 141 (1978), no. 3-4, 165–186.

[31] E. Fabes, O. Mendez and M. Mitrea, *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal., 159 (1998), 323–368.

[32] C. Fefferman, E.M. Stein, *H^p spaces of several variables*, Acta Math., 129 (1972), no. 3-4, 137–193.

[33] J. Franke, *On the spaces F_{pq}^s of Triebel-Lizorkin type: pointwise multipliers and spaces on domains*, Math. Nachr., 125 (1986), 29–68.

[34] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J., 34 (1985), 777–799.

[35] M. Frazier and B. Jawerth, *The ϕ-transform and applications to distribution spaces*, Function Spaces and Applications, pp. 223–246, Lecture Notes in Math., No. 1302, Springer, Berlin, 1988.

[36] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal., 93 (1990), no. 1, 34–170.

[37] M. Frazier and B. Jawerth, *Applications of the φ and wavelet transforms to the theory of function spaces*, Wavelets and Their Applications, pp. 377–417, Jones and Bartlett, Boston 1992.

[38] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, Vol. 79, AMS, Providence, RI, 1991.

[39] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J., 46 (1979), 27–42.

[40] M. Gomez and M. Milman, *Complex interpolation of H^p-spaces on product domains*, Ann. Math. Pura Appl., 155 (1989), 103–115.

[41] J. Janson and P.W. Jones, *Interpolation between H^p spaces: The complex method*, J. Funct. Anal., 48 (1982), 58–80.

[42] D. Jerison and C. Kenig, *The inhomogeneous problem in Lipschitz domains*, J. Funct. Anal., 130 (1995), 161–219.

[43] N.J. Kalton, *Convexity conditions for non-locally convex lattices*, Glasgow Math. J., 25 (1984), 141–152.

[44] N.J. Kalton, *Analytic functions in non-locally convex spaces*, Studia Math., 83 (1986), 275–303.

[45] N.J. Kalton, *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math., 84 (1986), 297–324.

[46] N.J. Kalton, *Analytic functions in non-locally convex spaces*, Studia Math., 83 (1986), no. 3, 275–303.

[47] N. Kalton and M. Mitrea, *Stability of Fredholm properties on interpolation scales of quasi-Banach spaces and applications*, Transactions of the Amer. Math. Soc., 350, (1998), 3837–3901.

[48] N. Kalton, N.T. Peck and J. Roberts, *An F-Space Sampler*, London Mat. Soc. Lecture Notes, Vol. 89, Cambridge University Press, 1984.

[49] R. Källström, *Maromorphic extensions of regular distributions*, Commun. Math. Phys., 126 (1989), 157–166.

[50] G.A. Kalyabin, *Theorems on extension, multipliers and diffeomorphisms for generalized Sobolev-Liouville classes in domains with Lipschitz boundary*, Trudy Mat. Inst. Steklov., 172 (1985), 173–186.

[51] C.E. Kenig, *Elliptic boundary value problems on Lipschitz domains*, Beijing Lectures in Harmonic Analysis, pp. 131–183, Ann. of Math. Stud., 112, Princeton Univ. Press, Princeton, NJ, 1986.

[52] C.E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, CBMS Regional Conference Series in Mathematics, Vol. 83, AMS, Providence, RI, 1994.

[53] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Springer-Verlag, Berlin Heidelberg, New York, 1977.

[54] J.-L. Lions, E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, Vol. 1, Travaux et Recherches Mathématiques, No. 17 Dunod, Paris 1968.

[55] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$*, Trans. Amer. Math. Soc., 315 (1989), no. 1, 69–87.

[56] S. Mayboroda and M. Mitrea, *Sharp estimates for Green potentials on non-smooth domains*, Math. Res. Lett., 11 (2004), no. 4, 481–492.

[57] S. Mayboroda and M. Mitrea, *The solution of the Chang-Krantz-Stein conjecture*, preprint (2006).

[58] O. Mendez and M. Mitrea, *The Banach envelope of Besov and Triebel-Lizorkin spaces and applications*, J. Four. Anal. Appl., 6 (2000), 503–531.

[59] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Advanced Mathematics, Vol. 37, Cambridge University Press, Cambridge, 1992.

[60] M. Milman and M. Mitrea, *Perturbation results for the real method of interpolation and applications*, preprint (2005).

[61] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem*, J. Funct. Anal., 176 (2000), no. 1, 1–79.

[62] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Hölder continuous metric tensors*, Comm. Partial Differential Equations, 25 (2000), 1487–1536.

[63] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: L^p, Hardy and Hölder type results*, Communications in Analysis and Geometry, 9 (2001), 369–421.

[64] A. Miyachi, *H^p spaces over open subsets of $\mathbb{R}^n$*, Studia Math., 95 (1990), no. 3, 205–228.

[65] A. Miyachi, *Hardy-Sobolev spaces and maximal functions*, J. Math. Soc. Japan, 42 (1990), no. 1, 73–90.

[66] A. Miyachi, *Extension theorems for real variable Hardy and Hardy-Sobolev spaces*, Harmonic Analysis (Sendai, 1990), pp. 170–182, ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991.

[67] A. Miyachi, *Extension theorems for the function spaces of DeVore and Sharpley*, Math. Japon., 38 (1993), 1033–1049.

[68] A. Miyachi, *On the extension properties of Triebel-Lizorkin spaces*, Hokkaido Math. J., 27 (1998), no. 2, 273–301.

[69] T. Muramatu, *On Besov spaces and Sobolev spaces of generalized functions definded on a general region*, Publ. Res. Inst. Math. Sci., 9 (1973/74), 325–396.

[70] J. Peetre, *New Thoughts on Besov Spaces*, Duke University Mathematics Series, No. 1, Mathematics Department, Duke University, Durham, NC, 1976.

[71] J. Pipher, G. Verchota, *The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains*, Amer. J. Math., 114 (1992), no. 5, 923–972.

[72] N.M. Rivière, *Interpolation Theory in s-Banach Spaces*, Dissertation, Univ. of Chicago, Chicago, IL, 1966.

[73] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Operators*, de Gruyter, Berlin, New York, 1996.

[74] V. Rychkov, *On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains*, J. London Math. Soc. (2), 60 (1999), no. 1, 237–257.

[75] A. Seeger, *A note on Triebel-Lizorkin spaces*, Approximation and Function Spaces, pp. 391–400, Banach Center Publ., No. 22, PWN, Warsaw, 1989.

[76] Z. Slodkowski, *A generalization of Vesentini and Wermer's theorems*, Rend. Sem. Mat. Univ. Padova, 75 (1986), 157–171.

[77] I. Ya.Šneiberg, *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled., 9 (1974), 214–229.

[78] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970.

[79] E. Straube, *Interpolation between Sobolev and between Lipschitz spaces of analytic functions on star-shaped domains*, Trans. Amer. Math. Soc., 316 (1989), no. 2, 653-671.

[80] M.E. Taylor, *Partial Differential Equations. I. Basic Theory*, Applied Mathematical Sciences, No. 115, Springer-Verlag, New York, 1996.

[81] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[82] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Berlin, 1983.

[83] H. Triebel, *Theory of Function Spaces. II*, Monographs in Mathematics, Vol. 84, Birkhäuser Verlag, Basel, 1992.

[84] H. Triebel, *Function spaces on Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers*, Rev. Mat. Complut., 15 (2002), 475–524.

[85] P. Turpin, *Convexités dans les espaces vectoriels topologiques generaux*, Dissertationes Math., Vol. 131, 1974.

[86] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), no. 3, 572–611.

[87] A. T. Vignati and M. Vignati, *Spectral theory and complex interpolation*, J. Funct. Anal., 80 (1988), 387–397.

[88] M. Zafran, *Spectral theory and interpolation of operators*, J. Funct. Anal., 36 (1980), no. 2, 185–204.

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