Math 2374	**Name (Print):** ______________
Spring 2009	**Student ID:** ______________
Midterm 1	**Section Number:** ______
February 25, 2009	**Teaching Assistant:** ______________
Time Limit: 1 hour	**Signature:** ______________

This exam contains 4 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	25 pts	
2	25 pts	
3	20 pts	
4	25 pts	
5	25 pts	
6	20 pts	
TOTAL	140 pts	

SKETCH OF THE SOLUTION
there might be some typos

1. (25 points) Find the equation of the plane that contains the lines given in parametric form by $\mathbf{c}_1(t) = (0, 5-t, 0)$ and $\mathbf{c}_2(t) = (1, 2+t, 3)$. Give your answer in the form $Ax+By+Cz+D = 0$.

   We can write the parametric equations of the lines as

   $$\mathbf{c}_1(t) = (0,5,0) + t\,(0,-1,0), \qquad \mathbf{c}_2(t) = (1,2,3) + t\,(0,1,0).$$

   It is clear that these lines are parallel. We can then pick one point from each of them and compute the vector that goes from one to the other. For instance

   $$\mathbf{v}_2 = (1,2,3) - (0,5,0) = (1,-3,3).$$

   The normal vector can be computed doing the cross product of the directional vector for one of the lines $\mathbf{v}_1 = (0,-1,0)$ by the vector $\mathbf{v}_2$:

   $$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 0 \\ 1 & -3 & 3 \end{vmatrix} = (-3,0,1).$$

   The equation of the plane is $\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0) = 0$ where $\mathbf{x}_0$ is a point of the plane. With $\mathbf{x}_0 = (0,5,0)$ we obtain

   $$(-3,0,1)\cdot(x-0, y-5, z-0) = 0$$

   that is

   $$-3\,x + z = 0.$$

2. (25 points) Consider the surface given by $g(x,y,z) = 7$ where $g(x,y,z) = x^2 - y^2 + \frac{z^2}{4}$.

   (a) (15 points) Show that the point $(2,1,4)$ lies on the surface and find the equation of the tangent plane to the surface at this point. (Give your answer in the form $Ax+By+Cz+D = 0$.)

   By checking that

   $$g(2,1,4) = 2^2 - 1^2 + \tfrac{4^2}{4} = 7$$

   we have proved that $(2,1,4)$ is on the surface. For the tangent plane we first need the gradient of g

   $$\nabla g(x,y,z) = \big(2x, -2y, \tfrac{z}{2}\big).$$

   Therefore $\nabla g(2,1,4) = (4,-2,2)$. The formula for the tangent plane at a point of a level surface of g gives

   $$(4,-2,2)\cdot(x-2, y-1, z-4) = 0,$$

   which can be simplified to

   $$4x - 2y + 2z - 14 = 0.$$

(b) (10 points) Find the points of the surface where the tangent plane is horizontal.

If the tangent plane is horizontal, the gradient has to have the form $(0, 0, c)$. Going back to the formula for the gradient at a general point (x, y, z), we see that necessarily $x = y = 0$. Plugging these values in the expression for the surface, we obtain

$$\frac{z^2}{4} = 7, \qquad \text{and therefore} \qquad z = \pm\sqrt{28}.$$

The tangent plane is horizontal at two points of the surface $(0, 0, \pm\sqrt{28})$.

3. (20 points) The trajectory of a flying mosquito is given by the path $\mathbf{c}(t) = (\cos t, t, \sin t)$. The temperature at each point of the space is measured by a function $T(x, y, z)$ and we know that $\nabla T(x, y, z) = (-z, y^2, x)$. Find the rate of change of temperature that the mosquito is experiencing at any given time t.

The temperature experienced by the mosquito at a time t is the function $T \circ \mathbf{c}(t)$. By the chain rule

$$\begin{aligned}(T \circ \mathbf{c})'(t) &= \nabla T(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \\ &= (-\sin t, t^2, \cos t) \cdot \mathbf{c}'(t) \\ &= (-\sin t, t^2, \cos t) \cdot (-\sin t, 1, \cos t) = (\sin t)^2 + t^2 + (\cos t)^2 = 1 + t^2.\end{aligned}$$

This is the rate of change of temperature experienced by the mosquito at time t. The chain rule can be written also using the matrix form of the derivatives (a row vector for T and a column vector for $\mathbf{c}$):

$$\mathbf{D}T(\mathbf{c}(t))\,\mathbf{D}\mathbf{c}(t) = \begin{bmatrix} -\sin t & t^2 & \cos t \end{bmatrix} \begin{bmatrix} -\sin t \\ 1 \\ \cos t \end{bmatrix}$$

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4. (25 points) Give a linear approximation of the function

$$f(x, y) = \left(2\,y + \frac{1}{2}\right) e^{2x-4+y^2}$$

near the point $(2, 0)$. Use it to approximate the value of $f(1.9,\, 0.01)$.

We start by computing

$$f(2, 0) = \tfrac{1}{2}\, e^0 = \tfrac{1}{2}.$$

The gradient of f is

$$\nabla f(x, y) = \left(2e^{2x-y+y^2}\,(2y + \tfrac{1}{2}),\; e^{2x-4+y^2}\,(2 + 4y^2 + y)\right)$$

which at the point $(2, 0)$ gives $\nabla f(2, 0) = (1, 2)$. The linear approximation to f near $(2, 0)$ is

$$z = \tfrac{1}{2} + (1, 2) \cdot (x - 2, y - 0) = \tfrac{1}{2} + (x - 1) + 2\,y = -\tfrac{3}{2} + x + 2\,y.$$

We can use to this expression to approximate

$$f(1.9,\, 0.01) \approx 0.5 + (-0.1) + 2\,(0.01) = 0.42.$$

5. (25 points) The height of land in a certain region is given as a function of the horizontal coordinates $h(x, y) = 3 + x^2 y + y^3$.

   (a) (15 points) If we are located in the point with horizontal coordinates $(1, 1)$, what is the direction of steepest descent at that point? (Give it as a unit vector). Compute the slope in that direction.

   The direction of steepest descent is that of $-\nabla h(1, 1)$. We first compute the gradient at a general point
$$\nabla h(x, y) = (2xy,\, x^2 + 3y^2).$$
   Evaluating at $(x, y) = (1, 1)$ we get $\nabla h(1, 1) = (2, 4)$. The norm of this vector (which is the same as the norm of $-\nabla h(1, 1)$) is
$$\|\nabla h(1, 1)\| = \|(2, 4)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}.$$
   The direction of steepest descent, given as a unit vector is
$$-\frac{1}{\|\nabla h(1, 1)\|}\,\nabla h(1, 1) = \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right).$$
   The slope in this direction is the directional derivative, which, by the way we choose the direction is simply $-\|\nabla h(1, 1)\| = -2\sqrt{5}$.

   (b) (10 points) Compute $\mathbf{D_v} h(1, 1)$ when $\mathbf{v}$ is the unit vector in the direction of $(-1, 2)$.

   We start by computing the norm of the vector $\|(-1, 2)\| = \sqrt{5}$. The vector is $(-1, 2)$ normalized, that is
$$\mathbf{v} = \frac{1}{\sqrt{5}}(-1, 2) = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$
   and the directional derivative is
$$\mathbf{D_v} h(1, 1) = \big(\nabla h(1, 1)\big) \cdot \mathbf{v} = (2, 4) \cdot \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{6}{\sqrt{5}}$$

6. (20 points) Compute the area of the triangle whose vertices are $(1, 1, 1)$, $(2, 1, 0)$ and $(0, 3, 2)$.

   We first pick one of the points $(1, 1, 1)$ and compute the vectors that go from this point to the other two points
$$\begin{aligned} \mathbf{v}_1 &= (2, 1, 0) - (1, 1, 1) = (1, 0, -1) \\ \mathbf{v}_2 &= (0, 3, 2) - (1, 1, 1) = (-1, 2, 1) \end{aligned}$$
   The area of the triangle is $\frac{1}{2}\|\mathbf{v}_1 \times \mathbf{v}_2\|$. We start by computing the cross product
$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{vmatrix} = 2\,\mathbf{i} + 0\,\mathbf{j} + 2\,\mathbf{k} = (2, 0, 2).$$
   Finally the area of the triangle is
$$\frac{1}{2}\,\|(2, 0, 2)\| = \frac{1}{2}\,\sqrt{4 + 0 + 4} = \sqrt{2}.$$