

Math 5285H

Midterm 1

No collaboration is allowed. This test is open-book and open-library but no electronic sources may be consulted.

This test is due in-class on **Friday, October 16**.

1. Suppose G is a group and $g \in G$. The *centralizer* of g is the set

$$C_g = \{h \in G \mid hgh^{-1} = g\}.$$

- (a) Show that the centralizer is a subgroup of G .
- (b) Construct a bijection between the set of conjugates of g and the set of left cosets G/C_g .
2. Let H and K be subgroups of a group G . Show that $|HK| = |H| \cdot |K|$ if and only if $H \cap K = \{e\}$.
3. Let Q_8 be the quaternion group with 8 elements (see the textbook, page 48). Find all the subgroups of Q_8 , and determine which are normal.
4. Find all solutions to the following equations in modular arithmetic.
- (a) $x^2 = \bar{1}$ in $\mathbb{Z}/7$.
- (b) $x^2 = \bar{1}$ in $\mathbb{Z}/16$.
- (c) $x^2 = \overline{-1}$ in $\mathbb{Z}/4$.
- (d) $x^2 = \overline{-1}$ in $\mathbb{Z}/13$.

(Here $x^2 = x * x$, using multiplication in modular arithmetic and *not* using multiplicative notation for the additive group law.)

5. Suppose H and K are subgroups of G . A *double coset* is a subset of G of the form

$$HgK = \{h g k \mid h \in H, k \in K\}$$

- (a) Prove or find a counterexample: These double cosets are a partition of G .
- (b) Prove or find a counterexample: If G is finite, all these double cosets have the same size.