Math 5285H

Midterm 2

No collaboration is allowed. This test is open-book and open-library but no electronic sources may be consulted.

This test is due in-class on **Friday, November 20**.

1. Prove that a real 3×3 matrix has at least one eigenvector.

2. Give a matrix expression for the linear operator on $\mathbb{R}^3$ that takes a vector v to its projection onto the line generated by the vector $(1, 2, -2)$. Find the rank of this matrix.

3. Let P_2 be the vector space of polynomials of degree 2 or less with coefficients in $\mathbb{R}$.

   (a) Show that the set $((x-1)^2, x^2, (x+1)^2)$ is a basis of P_2.

   (b) Let $D : P_2 \to P_2$ be the linear operator sending a polynomial to its derivative: $D(f(x)) = f'(x)$. Express D as a matrix in terms of the above basis.

4. Find all possible solutions to the following first-order linear differential equations:

   (a) $x' = y$, $y' = 2x + y$

   (b) $u' = 2u + 3v$, $v' = 2v$

5. Let $\mathbb{F} = \mathbb{Z}/2$ be the field with two elements, and consider the following two "dot products" on $\mathbb{F}^2$:

   - $(x, y) \cdot (z, w) = xz + yw$
   - $(x, y) * (z, w) = xw + yz$

   Show that there is no linear operator T on $\mathbb{F}^2$ such that $T(x, y) \cdot T(z, w) = (x, y) * (z, w)$.