Math 5378, Differential Geometry
Solutions to practice questions for Test 1

1. Find a parametrized curve whose trace is the set of points (x, y) in $\mathbb{R}^2$ with $xy = 1$, $x > 0$.

   Solution: One possible solution is $\alpha(t) = (t, 1/t)$ for $t \in (0, \infty)$. Remember to specify the interval that your curve is parametrized on.

2. Find the arc length of the curve

$$\alpha(t) = \left(t \sin t, t \cos t, \frac{\sqrt{8}}{3} t^{3/2} \right)$$

   between $t = 0$ and $t = 1$.

   Solution: The velocity is

$$\alpha'(t) = (\sin t + t \cos t, \cos t - t \sin t, \sqrt{2} t^{1/2}),$$

   and the speed simplifies to

$$\| \alpha'(t) \| = \sqrt{1 + t^2 + 2t} = (1 + t).$$

   Therefore, the length between $t = 0$ and $t = 1$ is

$$\int_0^1 (1 + t) dt = 3/2.$$

3. Show that the curve

$$\alpha(t) = (\sin t, t, -\cos t)$$

   has constant speed. Then find a reparametrization of this curve by arc length.

   Solution: The velocity is

$$\alpha(t) = (\cos t, 1, \sin t)$$

   and the speed simplifies to

$$\| \alpha'(t) \| = \sqrt{2},$$

   which is constant.

   Both $\beta_1(s) = \alpha(s/\sqrt{2})$ and $\beta_2(s) = \alpha(-s/\sqrt{2})$ are reparametrizations by arc length.

4. Give the Frenet formulas for the derivatives of the tangent, normal, and binormal of a curve parametrized by arc length.

   Solution: If t, n, and b are the tangent, normal, and binormal vectors, then the Frenet formulas say the following.

$$\begin{aligned} t'(s) &= k(s)n(s) \\ n'(s) &= -k(s)t(s) - \tau(s)b(s) \\ b'(s) &= \tau(s)n(s) \end{aligned}$$

   Here $k(s)$ is the curvature and $\tau(s)$ is the torsion. (Or you could write it in matrix form.)

5. If $\alpha(s)$ is a curve parametrized by arc length, prove that $\alpha''(s)$ is perpendicular to $\alpha'(s)$.

   Solution: If α is parametrized by arc length, then

$$1 = \| \alpha'(s) \|^2 = \alpha'(s) \cdot \alpha'(s).$$

   Differentiating this by using the Leibniz rule, we find

$$0 = \alpha''(s) \cdot \alpha'(s) + \alpha'(s) \cdot \alpha''(s),$$

   or equivalently

$$0 = 2\alpha''(s) \cdot \alpha'(s).$$

   Since these vectors have dot product equal to zero, they are perpendicular.

6. Find all real numbers c so that the set

$$\{(x, y, z) \mid x^2 - y^2 + z^3 - z = c\}$$

   is a smooth surface in $\mathbb{R}^3$.

   Solution: This set is the set of zeros of the function

$$f(x, y, z) = x^2 - y^2 + z^3 - z - c.$$

   The gradient of this function is

$$\nabla f = [2x, -2y, 3z^2 - 1],$$

which vanishes precisely when $x = 0$, $y = 0$, and $z = \pm\sqrt{1/3}$. However, we only have $f(0, 0, \sqrt{1/3}) = 0$ when $c = \frac{-2}{3\sqrt{3}}$, and we only have $f(0, 0, -\sqrt{1/3}) = 0$ when $c = \frac{2}{3\sqrt{3}}$.

Therefore, this equation defines a smooth surface whenever $c \neq \pm\frac{2}{3\sqrt{3}}$.

7. Consider the coordinate chart

$$\mathbf{x}(u, v) = (u^2, u^3 + v^3, v^2).$$

for $u, v > 0$. Find the coefficients E, F, and G of the first fundamental form in these coordinates.

Solution: The tangent vectors are

$$\mathbf{x}_u = (2u, 3u^2, 0), \mathbf{x}_v = (0, 3v^2, 2v).$$

Therefore, the coefficients of the first fundamental form are the following.

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 4u^2 + 9u^4 \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 9u^2v^2 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 9v^4 + 4v^2 \end{aligned}$$

8. Suppose we have a coordinate chart $\mathbf{x}$ on the open set

$$\{(u, v) \in \mathbb{R}^2 \mid u < 0, -\pi < v < \pi\}$$

such that the coefficients of the first fundamental form are:

- $E = e^u$,
- $F = 0$,
- $G = e^u$.

Find the length of the image of the curve $\alpha(t) = (-1, t)$ between $t = 0$ and $t = 1$.

Solution: The tangent vector is $\alpha'(t) = (0, 1)$, and the length of the image vector is calculated using the first fundamental form:

$$\begin{aligned} \| \alpha'(t) \|^2 &= E_{\alpha(t)}(0)^2 + 2F_{\alpha(t)}(0)(1) + G_{\alpha(t)}(1)^2 \\ &= e^{-1}. \end{aligned}$$

Therefore, the length of the image is

$$\int_{-1}^{0} \sqrt{e^{-1}}dt = \sqrt{e^{-1}}.$$

(Note a correction; I forgot the square root in the original version of this solution set.)

9. With the same coordinate chart as the previous problem, find the area of the image of the entire region under $\mathbf{x}$.

   Solution: The area of the image is the integral

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{-\infty}^{0} \sqrt{EG - F^2}\, du\, dv &= \int_{-\pi}^{\pi}\int_{-\infty}^{0} e^u\, du\, dv \\ &= \int_{-\pi}^{\pi} dv \\ &= 2\pi.\end{aligned}$$

10. Consider the coordinate chart

$$\mathbf{x}(u, v) = (u, u^2 + v^2, -v).$$

   Find a field N of unit normal vectors for this coordinate chart.

   Solution: The tangent vectors to the surface are $\mathbf{x}_u = (1, 2u, 0)$ and $\mathbf{x}_v = (0, 2v, -1)$, and so a unit normal vector field is given by

$$\begin{aligned}\frac{\mathbf{x}_u \times \mathbf{x}_v}{\| \mathbf{x}_u \times \mathbf{x}_v \|} &= \frac{(-2u, 1, 2v)}{\| (-2u, 1, 2v) \|} \\ &= \left(\frac{-2u}{\sqrt{4u^2 + 4v^2 + 1}}, \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}, \frac{2v}{\sqrt{4u^2 + 4v^2 + 1}}, \right).\end{aligned}$$

   You could equally well have the negative of this vector field instead.

11. Given a surface S with unit normal vector field N, give the mathematical definition of the second fundamental form $\mathrm{II}_p(v)$ for a vector v in the tangent space $T_p(S)$.

   Solution: The second fundamental form is

$$\mathrm{II}_p(v) = -\langle dN_p v, v \rangle,$$

   where dN_p is the differential of the function N at the point p.

12. Prove that a point p of a smooth surface is umbilical if and only if the Gaussian curvature K and the mean curvature H satisfy $H^2 = K$.

Solution: Recall that a point is umbilical if and and only if the two principal curvatures k_1 and k_2 are equal.

If $k_1 = k_2$, then $K = k_1 k_2 = k_1^2$ and $H = \frac{k_1+k_2}{2} = k_1$, so $H^2 = K$.

Conversely, if $H^2 = K$, then

$$\begin{aligned} k_1 k_2 &= \left(\frac{k_1 + k_2}{2}\right)^2 \\ &= \frac{1}{4}k_1^2 + \frac{1}{2}k_1 k_2 + \frac{1}{4}k_2^2. \end{aligned}$$

Subtracting $k_1 k_2$ from both sides, we get

$$\begin{aligned} 0 &= \frac{1}{4}k_1^2 - \frac{1}{2}k_1 k_2 + \frac{1}{4}k_2^2 \\ &= \left(\frac{k_1 - k_2}{2}\right)^2. \end{aligned}$$

If this number squares to zero, then it must be equal to zero, so $k_1 = k_2$.

Note, it is important to show both directions:

- if the principal curvatures are equal, then $K = H^2$, and
- if $K = H^2$, then the principal curvatures are equal.