Math 5378, Differential Geometry
Solutions to practice questions for Test 2

1. Find all possible trajectories of the vector field $w(x,y) = (-y,x)$ on $\mathbb{R}^2$.

   Solution: A trajectory would be a curve $(x(t), y(t))$ satisfying $x' = -y, y' = x$, and hence $x'' = -x$. Therefore, we would have $x(t) = a\cos(t) + b\sin(t)$ for some constants a, b, and $y = -x' = a\sin(t) - b\cos(t)$.

2. If the first fundamental form in coordinates is given by $E = e^u, F = 0, G = e^v$, find a vector field of unit length perpendicular to the vector field $x_u - x_v$.

   Solution: We apply the first fundamental form. For a vector field $ax_u + bx_v$ to be perpendicular, we require:

   $$0 = \langle x_u - x_v, ax_u + bx_v \rangle = ae^u - be^v,$$

   so $b = ae^{u-v}$. To be unit length, we require:

   $$1 = ||ax_u + ae^{u-v}x_v||^2 = a^2e^u + a^2e^{2u-2v}e^v,$$

   so $a = (e^u + e^{2u-v})^{-1/2}$. Plugging back in gives us the desired vector field.

3. If $f : S_1 \to S_2$ is an isometry between surfaces and $\alpha(s) : (a,b) \to S_1$ is a geodesic parametrized by arc length, show that $f(\alpha(s))$ is also a geodesic parametrized by arc length.

   Solution: An isometry preserves the first fundamental form, and hence the lengths of vectors, so $f(\alpha(s))$ has the same speed as $\alpha(s)$. Moreover, we have shown that the Christoffel symbols in coordinates depend only on the first fundamental form, and an isometry preserves the first fundamental form, so the requirements that $\alpha(s)$ and $f(\alpha(s))$ be geodesics are given by the same differential equations (which we state in a later problem).

4. Suppose $\mathbf{x}$ is a coordinate chart on a surface, with coefficients E, F, and G of the first fundamental form. Prove the following identities.

$$\begin{aligned}\langle x_{uu}, x_u \rangle &= \frac{1}{2} E_u \\ \langle x_{uu}, x_v \rangle &= F_u - \frac{1}{2} E_v\end{aligned}$$

Use these to show the matrix identity

$$\begin{bmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix}$$

Solution: We recall that by definition, $E = \langle x_u, x_u \rangle$, $F = \langle x_u, x_v \rangle$, so their derivatives (by the Leibniz rule) are given by

$$\begin{aligned}E_u &= 2\langle x_{uu}, x_u \rangle, \\ E_v &= 2\langle x_{uv}, x_u \rangle, \\ F_u &= \langle x_{uu}, x_v \rangle + \langle x_u, x_{uv} \rangle.\end{aligned}$$

Solving produces the first required formulas.

To get the matrix equation, we take the equation

$$x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + eN$$

and take dot products with x_u and x_v (both perpendicular to N) to get formulas as follows:

$$\begin{aligned}\langle x_{uu}, x_u \rangle &= \Gamma_{11}^1 E + \Gamma_{11}^2 F \\ \langle x_{uu}, x_v \rangle &= \Gamma_{11}^1 F + \Gamma_{11}^2 G\end{aligned}$$

Plugging in for the inner products and rewriting this in matrix form gives the second desired formula.

5. Prove that the sphere of radius $R > 0$ centered at the origin has constant Gaussian curvature $1/R^2$ and mean curvature $1/R$.

Solution: A normal vector field on this sphere is given by $N(x, y, z) = (x/R, y/R, z/R)$ (which is a unit normal vector), or $N(v) = \frac{1}{R}v$, which is just scalar multiplication. The differential dN then is given by $dN(w) = \frac{1}{R}w$ for any tangent vector w, and so the matrix of dN is $\frac{1}{R}$ times the identity matrix.

The Gaussian curvature is the determinant of this matrix, which is $1/R^2$. The trace of this matrix is $2/R$; dividing this by 2 and making this negative gives the mean curvature, which is $-1/R$.

6. Suppose $(u(s), v(s))$ is a curve in $\mathbb{R}^2$ and $\mathbf{x}$ is a coordinate chart so that $\mathbf{x}(u(s), v(s))$ is a curve parametrized by arc length. Write down the conditions on u and v necessary for this curve to be a geodesic in the surface.

   Solution: The differential equations of a constant speed geodesic are:

$$\begin{aligned} u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}(u'v') + \Gamma^1_{22}(v')^2 &= 0, \\ v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}(u'v') + \Gamma^2_{22}(v')^2 &= 0. \end{aligned}$$

7. Let $\alpha(s) = (f(s), g(s))$ be a curve in $\mathbb{R}^2$ parametrized by arc length, and consider the coordinate chart on the associated surface of revolution given by

$$\mathbf{x}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

   Prove that for any fixed angle θ, the meridian

$$\alpha(s) = (f(s)\cos\theta, f(s)\sin\theta, g(s))$$

   is a geodesic parametrized by arc length.

   Solution: To show this is parametrized by arc length, we calculate

$$\alpha'(s) = (f'(s)\cos\theta, f'(s)\sin\theta, g'(s)).$$

   This has length $\sqrt{(f'(s))^2 + (g'(s))^2}$, which is 1 because the original curve was parametrized by arc length.

   One can find the first fundamental form, Christoffel symbols, and the covariant derivative explicitly. However, it is easier to simply note that the covariant derivative is the projection of the second derivative onto the tangent space. The second derivative here is

$$\alpha''(s) = (f''(s)\cos\theta, f''(s)\sin\theta, g''(s))$$

and the tangent vectors at the point $\alpha(s)$ are

$$\begin{aligned} x_u &= (f'(s)\cos\theta, f'(s)\sin\theta, g'(s)), \\ x_v &= (-f(s)\sin\theta, f(s)\cos\theta, 0). \end{aligned}$$

The first coincides with $\alpha'(s)$, which is perpendicular to $\alpha''(s)$ because α is parametrized by arc length. The second we can see is perpendicular to $\alpha''(s)$ by direct calculation.

Therefore, since $\alpha''(s)$ is perpendicular to the tangent space, the covariant derivative $D\alpha'/ds$ is zero.

8. Explain the sequence of steps (without calculating anything) taken to derive the Mainardi-Codazzi equations relating Christoffel symbols to e, f, and g from the formulas for x_{uu}, x_{uv}, and x_{vv}.

   Solution: This is a vague question, but one basic idea is the following:

   - We start with the equations

   $$\begin{aligned} x_{uu} &= \Gamma^1_{11}x_u + \Gamma^2_{11}x_v + eN, \\ x_{uv} &= \Gamma^1_{12}x_u + \Gamma^2_{12}x_v + fN. \end{aligned}$$

   - We apply the identity $(x_{uu})_v = (x_{uv})_u$, and plug these equations into both sides.
   - We take the dot product with the unit normal vector N (or equivalently ignore the x_u and x_v components of the result) remembering that N_u and N_v are perpendicular to N.

   The resulting equation is one of the Mainardi-Codazzi equations; we get the other one by looking at $(x_{uv})_v = (x_{vv})_u$ and comparing normal components.

9. Find the absolute value of the geodesic curvature of the curve $(\cos t\cos\theta, \sin t\cos\theta, \sin\theta)$ on S^2 for any fixed value of θ.

   Solution: We first note that this curve moves at speed $\cos\theta$, and so we reparametrize by arc length as

   $$\beta(s) = (\cos(s/\cos\theta)\cos\theta, \sin(s/\cos\theta)\cos\theta, \sin\theta).$$

This curve has tangent vector

$$\beta'(s) = (-\sin(s/\cos\theta), \cos(s/\cos\theta), 0),$$

and second derivative

$$\beta''(s) = (-\cos(s/\cos\theta)/\cos\theta, -\sin(s/\cos\theta)/\cos\theta, 0),$$

The length $1/\cos\theta$ of this vector is the curvature k. The unit normal vector at $\beta(s)$ is $\beta(s)$, and so the length of the normal component is the normal curvature

$$k_n = |\beta(s) \cdot \beta''(s)| = |-\cos^2(s/\cos\theta) - \sin^2(s/\cos\theta)| = 1.$$

Since the geodesic and normal curvatures satisfy $k^2 = k_n^2 + k_g^2$, we find

$$|k_g| = \sqrt{\frac{1}{\cos\theta} - 1}.$$

10. On a sphere of radius $R > 0$, suppose that we have a triangle with three geodesic sides, with interior angles θ_1, θ_2, and θ_3. Find the area of the triangle.

    Solution: The sphere has constant Gaussian curvature $1/R^2$. Therefore, applying the Gauss-Bonnet theorem to this triangle, we find

    $$Area(T)/R^2 = \theta_1 + \theta_2 + \theta_3 - \pi,$$

    or

    $$Area(T) = R^2(\theta_1 + \theta_2 + \theta_3 - \pi).$$

11. Show that on a surface of nonpositive curvature, there are no simple closed geodesics that bound simple regions.

    Solution: If such a simple closed geodesic existed bounding a simple region R (which has Euler characteristic 1) then both the geodesic curvature and angle terms would vanish from the equation in the Gauss-Bonnet theorem. We would find

    $$0 \geq \int\int_R K = 2\pi,$$

    a contradiction.

12. Calculate the geodesic curvature of the circle $z = h$ on the cone $x^2 + y^2 = z^2$. Explain how the Gauss-Bonnet theorem relates these for different values of h.

Solution: We can parametrize this curve by arc length as

$$\beta(s) = (h\cos(s/h), h\sin(s/h), h).$$

Then we can calculate as in the previous problem.

$$\beta'(s) = (-\sin(s/h), \cos(s/h), 0)$$

$$\beta''(s) = (-\cos(s/h)/h, -\sin(s/h)/h, 0)$$

The curvature is $k = |\beta''(s)| = 1/h$. The normal vector at $\beta(s)$ is $\frac{1}{\sqrt{2}}(\cos(s/h), \sin(s/h), -1)$, obtained by normalizing the gradient vector. Therefore,

$$k_n = \frac{1}{\sqrt{2}}|\beta''(s) \cdot (\cos(s/h), \sin(s/h), -1)| = \frac{1}{h\sqrt{2}}.$$

As a result, the geodesic curvature is $k_g = \sqrt{k^2 - k_n^2} = \frac{1}{h\sqrt{2}}$.

Given two different heights h_1 and h_2, they enclose an annulus of Euler characteristic zero. The cone has geodesic curvature 0, so the Gauss-Bonnet theorem says that for any two of these curves C_1 and C_2 at heights h_1 and h_2 respectively,

$$\int_{C_1} \frac{1}{h_1\sqrt{2}} = \int_{C_2} \frac{1}{h_2\sqrt{2}}.$$

(This is true because the circle at height h has circumference proportional to h.)

13. Calculate the index of the critical point $(0, 0)$ of the vector field

$$w(x, y) = (x^2 - y^2, 2xy)$$

on $\mathbb{R}^2$.

Solution: We first note that this is the only singularity in all of $\mathbb{R}^2$.

We parametrize the circle $x^2 + y^2 = 1$ as $(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then in these coordinates, we find the following.

$$\begin{aligned} w(x, y) &= (\cos^2 t - \sin^2 t, 2\cos t \sin t) \\ &= (\cos(2t), \sin(2t)). \end{aligned}$$

In the interval $[0, 2\pi]$, this vector field rotates by an angle of 4π, and therefore the index is 2.