Math 8253, Fall 2015
Homework 2
Due in-class on **Monday, September 28**

1. Show that, for any irreducible $X \subset \mathbb{A}^n$, the closure $\bar{X}$ of X in $\mathbb{P}^n$ is irreducible. Determine the closures in $\mathbb{P}^2$ of the plane curves
$$\{(x, y) \mid y^2 = x^3 + ax^2 + bx + c\}$$
and
$$\{(x, y) \mid x^2 + bxy + cy^2 + dx + ey + f = 0\}.$$

2. Suppose that an ideal $J \subset k[x_1, \ldots, x_n]$ is contained in the maximal ideal $M = (x_1, \ldots, x_n)$ and that J can be generated by r elements $f_1, \ldots, f_r$.

   Let M^2 be the ideal generated by elements $x_i x_j$ for $1 \leq i, j \leq n$. Show that the dimension of the ring
$$k[x_1, \ldots, x_n]/(J + M^2),$$
viewed as a k-vector space, is at least $1 + n - r$.

3. Using the previous problem, show that the curve from the previous problem set, defined by the equations
$$xz = y^2, x^3 = yz, z^2 = x^2 y,$$
cannot be the solution set of any collection with less than three equations.

4. Define a map $p : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, given by
$$p(x_0, x_1, \ldots, x_n) = [x_0 : x_1 : \cdots : x_n].$$
Show that this map is algebraic by showing that its restrictions to the affine subvarieties $\mathbb{A}^{n+1} \setminus \{x_i = 0\}$ are algebraic.

5. Suppose we have the nondegenerate conic C in $\mathbb{P}^2$ defined by
$$\{[x : y : z] \mid x^2 - y^2 = z^2\},$$
so that $[1 : 0 : 1]$ is a solution. Show that the map, sending a point $[x : y : 1]$ of C to the slope $[y : x - 1]$ of the line through (x, y) and $(1, 0)$, extends to an isomorphism of varieties between C and $\mathbb{P}^1$. (This construction works for all nondegenerate conics with a chosen point.)