Math 8301, Manifolds and Topology
Homework 7
Due in-class on **Monday, Oct 29**

1. (CORRECTED) Let $F = \langle x, y\rangle$ be a free group on two generators. Show explicitly that the subgroup $H \subset F$ generated by the elements $z_n = (y^{-n}xy^n)$, as n ranges over the integers, is a free group on the elements z_n.

2. If F and F' are free groups, show that $F * F'$ is also free.

3. Suppose $f : H \to G$ is a group homomorphism. Show by universal property that the amalgamated product $G *_H \{e\}$ is always isomorphic to the quotient G/N, where N is the normal subgroup generated by the image of f.

4. Suppose X is a topological space and $\phi : S^{n-1} \to X$ is a continuous function, where $n \geq 1$. Let

   $$Y = X \cup_\phi e^n$$

   be the space obtained from the disjoint union $D^n \cup X$ by identifying any point $v \in S^{n-1} \subset D^n$ with its image $\phi(v) \in X$. We say that Y is obtained from X by attaching an n-cell.

   Use the Seifert-van Kampen theorem to describe the fundamental group of Y in terms of the fundamental group of X and the map ϕ in the cases $n > 1$.

5. Suppose X is path-connected and p, q are points in X, which determine a map $S^0 \to X$. Show that the fundamental group of $X \cup_\phi e^1$ at p is the free product $\pi_1(X, p) * \mathbb{Z}$. (Hint: Seifert-van Kampen is hard to use directly here. Start by finding a loop $S^1 \to X \cup_\phi e^1$ and construct a homotopy equivalent space by gluing in $S^1 \times [0, 1]$ along $S^1 \times \{0\}$.)