Math 8302, Manifolds and Topology II
Homework 4
Due in-class on **Monday, March 4**

1. For this problem, you may assume (from class) that there is a smooth function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

   (a) $f(x) = 0$ for $x \leq 0$

   (b) $f(x) = 1$ for $x \geq 1$

   (c) $f'(x) > 0$ for $0 < x < 1$

   Suppose $(a_n)_{n \in \mathbb{Z}}$ is a list of real numbers (CORRECTION) such that $a_n \neq a_{n+1}$. Construct a smooth function $g(x) : \mathbb{R} \to \mathbb{R}$ whose singular points consist precisely of the integers, and so that for all n the singular value $g(n)$ is equal to a_n.

2. Give an example of a regularly embedded submanifold $M \to N$ which is not a fiber of a smooth map: that is, there is no smooth function $g : N \to P$ and point $p \in P$ with $M = g^{-1}(p)$.

3. Suppose M and N are smooth manifolds, $f : M \to N$ is a *regular embedding*, and X is a smooth vector field on N with the following property: *For all $p \in M$, the vector $X(f(p))$ is in the image of the map $df_p : T_p(M) \to T_{f(p)}(N)$.*

   Show that this determines a *smooth* vector field $\tilde{X}$ on M such that $df_p(\tilde{X}(p)) = X(f(p))$ for all $p \in M$.

4. Suppose $f : M \to N$ is an embedding (but not necessarily a regular embedding) and X is a vector field on M. Does there necessarily exist a vector field X' on N such that $df_p(X(p)) = X'(f(p))$ for all $p \in M$? Either prove this statement or give a counterexample.

5. Suppose we have two vector fields on $\mathbb{R}^n$ of the form $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^i \frac{\partial}{\partial x^i}$ (where a^i and b^i are smooth functions on $\mathbb{R}^n$). Viewing these as operators $X, Y : C^\infty(M) \to C^\infty(M)$, find necessary and sufficient conditions for the identity $X \circ Y = Y \circ X$ to hold.