18.704 Problem Set 1 Solutions

1. The subgroups of Q_8 are:

$$\{1\}$$
$$\{1,-1\}$$
$$\{1,i,-1,-i\}$$
$$\{1,j,-1,-j\}$$
$$\{1,k,-1,-k\}$$
$$Q_8$$

The commutator subgroup contains the element

$$[i,j] = iji^{-1}j^{-1} = ij(-i)(-j) = (ij)(ij) = k^2 = -1.$$

Similarly $[j,k] = -1$ and $[k,i] = -1$. On the other hand, -1 and 1 commute with all elements of Q_8, so $[x,-1] = [x,1] = 1$ for all $x \in Q_8$.

Therefore, the commutator subgroup is the subgroup of Q_8 generated by -1 and 1, which is $\{1,-1\}$.

2. Since the group $\mathbb{C}^\times$ is abelian, any homomorphism $f : Q_8 \to \mathbb{C}^\times$ must send the commutator $-1 = [i,j]$ to 1.

We then must have

$$f(i)^2 = f(i^2) = f(-1) = 1$$

because f is a homomorphism. Similarly, $f(j)^2 = f(k)^2 = 1$. Therefore, f must take i, j, and k to ± 1.

Finally, we must have

$$f(k) = f(ij) = f(i)f(j).$$

Therefore, the only homomorphisms $Q_8 \to \mathbb{C}^\times$ are the following.

$$\begin{array}{llll} f(\pm 1) = 1, & f(\pm i) = 1, & f(\pm j) = 1, & f(\pm k) = 1 \\ f(\pm 1) = 1, & f(\pm i) = -1, & f(\pm j) = -1, & f(\pm k) = 1 \\ f(\pm 1) = 1, & f(\pm i) = -1, & f(\pm j) = 1, & f(\pm k) = -1 \\ f(\pm 1) = 1, & f(\pm i) = 1, & f(\pm j) = -1, & f(\pm k) = -1 \end{array}$$

3. The order of a double coset HxK does not always divide the size of the group.

Let $G = S_3$, the symmetric group on 3 letters, $H = \{e,(12)\}$, $x = e$, and $K = \{e,(23)\}$. The elements of the double coset are as follows.

$$\begin{aligned} HK &= \{hk \mid h \in H, k \in K\} \\ &= \{e \cdot e, e \cdot (23), (12) \cdot e, (12) \cdot (23)\} \\ &= \{e, (23), (12), (123)\} \end{aligned}$$

This double coset has 4 elements, but G has 6 elements.