18.704 Problem Set 2 Solutions

1. Proposition 4 states that if $\rho : G \to GL(V)$ is an irreducible representation, any linear transformation $T : V \to V$ such that $\rho(g)T = T\rho(g)$ is a homothety, i.e. $T = \lambda I$ for some $\lambda \in \mathbb{C}$. This is fails for representations over $\mathbb{R}$. The reason is that in the proof, we need to find an eigenvalue λ for the linear transformation T, and matrices in $GL_n(\mathbb{R})$ don't necessarily have real eigenvalues.

   Here is an example where it fails: Let C_4 be the cyclic group with 4 elements $\{1, x, x^2, x^3\}$, and let $\rho : C_4 \to GL_2(\mathbb{R})$ be the representation

   $$\rho(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   In this representation, x^k represents rotation by $k\cdot 90^o$. A nontrivial invariant subspace of this representation would have to be 1-dimensional, but no lines are preserved by rotation by 90^o. Therefore, this is an irreducible real representation.

   However, $T = \rho(x)$ satisfies $T\rho(g) = \rho(g)T$ for all $g \in G$, and T is a rotation, not a homothety.

2. (TEX question) Here is some example code.

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{\bf Proposition 3.} {\it Let $\rho: G \to {\rm GL}(V)$ be a
  linear representation of $G$, and let $\chi$ be its character.
  Let $\chi_\sigma^2$ be the character of the symmetric square
  ${\rm Sym}^2(V)$ of $V$ {\rm (cf 1.6)}, and let $\chi_\alpha^2$
  be that   of ${\rm Alt}^2(V)$.  For each $s \in G$, we have
$$
\chi_\sigma^2(s) = {1 \over 2}(\chi(s)^2 + \chi(s^2))
$$
$$
\chi_\alpha^2(s) = {1 \over 2}(\chi(s)^2 - \chi(s^2))
$$
and $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$.}

Let $s \in G$.  A basis $(e_i)$ of $V$ can be chosen consisting of
{\it eigenvectors} for $\rho_s$; this follows for example from the
fact that $\rho_s$ can be represented by a {\it unitary} matrix,
cf. 1.3.  We have then $\rho_s e_i = \lambda_i e_i$ with $\lambda_i
\in {\bf C}$, and so
$$
\chi(s) = \sum \lambda_i,\hskip 2pc \chi(s^2) = \sum \lambda_i^2.
$$
On the other hand, we have
$$
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(\rho_s \otimes \rho_s)(e_i \cdot e_j + e_j \cdot e_i) = \lambda_i
\lambda_j \cdot (e_i \cdot e_j + e_j \cdot e_i),
$$
$$
(\rho_s \otimes \rho_s)(e_i \cdot e_j - e_j \cdot e_i) = \lambda_i
\lambda_j \cdot (e_i \cdot e_j - e_j \cdot e_i),
$$
hence
$$
\chi_\sigma^2(s) = \sum_{i \leq j} \lambda_i \lambda_j = \sum
\lambda_i^2 + \sum_{i < j} \lambda_i \lambda_j = {1 \over 2}(\sum
\lambda_i)^2 + {1 \over 2}\sum \lambda_i^2
$$
$$
\chi_\alpha^2(s) = \sum_{i < j} \lambda_i \lambda_j =
{1 \over 2}(\sum \lambda_i)^2 - {1 \over 2}\sum \lambda_i^2
$$
The proposition follows.\hfill $\square$
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3. If χ and λ are both class functions on G, we have

$$
\begin{aligned}
(\chi * \lambda)(hgh^{-1}) &= \frac{1}{|G|}\sum_{x\in G}\chi(hgh^{-1}x^{-1})\lambda(x) \\
&= \frac{1}{|G|}\sum_{x\in G}\chi\Big(h^{-1}(hgh^{-1}x^{-1})h\Big)\lambda(x) \\
&= \frac{1}{|G|}\sum_{x\in G}\chi\Big(gh^{-1}x^{-1}h\Big)\lambda(x).
\end{aligned}
$$

Substituting $y = h^{-1}xh$ in the summation, we find

$$
\begin{aligned}
(\chi * \lambda)(hgh^{-1}) &= \frac{1}{|G|}\sum_{y\in G}\chi(gy^{-1})\lambda(hyh^{-1}) \\
&= \frac{1}{|G|}\sum_{y\in G}\chi(gy^{-1})\lambda(y) \\
&= (\chi * \lambda)(g)
\end{aligned}
$$

Therefore, $\chi * \lambda$ is a class function, as desired.