18.704 Problem Set 3 Solutions

1. $\hat{G}$ is an abelian group, with identity element $\mathbf{1}$ (the character of the trivial representation), as follows. For all $g \in G$ we have the following.

   - $(\mathbf{1} \cdot \chi)(g) = \mathbf{1}(g)\chi(g) = 1\chi(g) = \chi(g)$, so $\mathbf{1} \cdot \chi = \chi$.
   - $[\chi_1 \cdot (\chi_2 \cdot \chi_3)](g) = \chi_1(g)\chi_2(g)\chi_3(g) = [(\chi_1 \cdot \chi_2) \cdot \chi_3](g)$, so $\chi_1(\chi_2\chi_3) = (\chi_1\chi_2)\chi_3$.
   - $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g) = \chi_2(g)\chi_1(g) = (\chi_2 \cdot \chi_1)(g)$, so $\chi_1\chi_2 = \chi_2\chi_1$.
   - If χ is a character, so is its complex conjugate $\bar{\chi}$, and $(\chi \cdot \bar{\chi})(g) = |\chi(g)|^2 = 1 = \mathbf{1}(g)$ since $\chi(g)$ is a root of unity, so $\chi\bar{\chi} = \mathbf{1}$.

   For any $x \in G$, define $ev_x : \hat{G} \to \mathbb{C}^\times$ by $ev_x(\chi) = \chi(x)$. The map ev_x is a group homomorphism, as follows.

   - $ev_x(\mathbf{1}) = \mathbf{1}(x) = 1$.
   - $ev_x(\chi_1 \cdot \chi_2) = (\chi_1 \cdot \chi_2)(x) = \chi_1(x)\chi_2(x) = ev_x(\chi_1)ev_x(\chi_2)$.

   Therefore, ev_x is an element of $\hat{\hat{G}}$ for any $x \in G$.

   The map $ev : G \to \hat{\hat{G}}$ is an injective group homomorphism, as follows. (I am too lazy to figure out how to do a double-hat nicely today.)

   - $ev_1(\chi) = \chi(1) = 1$, so ev_1 is the trivial character of $\hat{G}$.
   - $ev_{xy}(\chi) = \chi(xy) = \chi(x)\chi(y) = ev_x(\chi)ev_y(\chi) = (ev_x \cdot ev_y)(\chi)$, so as characters of $\hat{G}$ we have $ev_x \cdot ev_y = ev_{xy}$.
   - If $ev_x = 1$, then $1 = ev_x(\chi) = \chi(x)$ for all χ, which would imply $x = 1$.

   Since G is abelian, and the number of irreducible representations of G is equal to the number of conjugacy classes of G, $\hat{G}$ has the same size as G. Similarly, $\hat{\hat{G}}$ has the same size as $\hat{G}$, so the map $G \to \hat{\hat{G}}$ is an isomorphism.

2. The group G of order 20

   $$\langle x, y \mid x^5 = y^4 = e, yxy^{-1} = x^2 \rangle$$

   has the following conjugacy classes.

   $$\{e\}, \{x, x^2, x^3, x^4\}, \{yx^k\}, \{y^2x^k\}, \{y^3x^k\}$$

   We also have $[y, x] = yxy^{-1}x^{-1} = x$, so the commutator subgroup contains x; in fact, it equals $< x >$. The subgroup generated by x is normal and the quotient $G/ < x >$ is isomorphic to $\mathbb{Z}/4$.

   Therefore, there are 5 irreducible representations, 4 of which are 1-dimensional. If the last one is d-dimensional, we must have $20 = 1^2 + 1^2 + 1^2 + 1^2 + d^2$, so $d = 4$.

The four 1-dimensional representations are given by

$$x^n y^m \mapsto e^{2\pi i k m/4}$$

for $k = 0, 1, 2, 3$. This gives us the first 4 rows of the character table. The last row we can cheat and get for free by using the character orthogonality relations.

	e	x^k	yx^k	y^2x^k	y^3x^k
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4	-1	0	0	0

3. Suppose that G acts on X doubly transitively, with permutation character χ_X.

$$\begin{aligned}\langle \chi_X, \mathbf{1}\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi(g) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{x\in X\\ gx=x}} 1 \\ &= \frac{1}{|G|}\sum_{x\in X}\sum_{\substack{g\in G\\ gx=x}} 1 \\ &= \frac{1}{|G|}\sum_{x\in X}|I_x|\end{aligned}$$

Here I_x is the stabilizer of X. Since

$$I_{gx} = gI_xg^{-1},$$

and the action of G is transitive on X, $|I_x|$ is the same for all $x \in X$. We then get the following.

$$\begin{aligned}\langle \chi_X, \mathbf{1}\rangle &= \frac{1}{|G|}|I_x|\sum_{x\in X} 1 \\ &= \frac{1}{|G|}|I_x||X| \\ &= 1\end{aligned}$$

$$
\begin{aligned}
\langle \chi_X, \chi_X \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g)^2 \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{\substack{x \in X \\ gx = x}} 1 \right)^2 \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{x \in X \\ gx = x}} \sum_{\substack{y \in X \\ gy = y}} 1 \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{(x,y) \in X \times X \\ (gx,gy) = (x,y)}} 1 \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{X \times X}(g) \\
&= \langle \chi_{X \times X}, \mathbf{1} \rangle
\end{aligned}
$$

Here $\chi_{X \times X}$ is the character of G acting on $X \times X$.

We have

$$X \times X = \{(x,x) | x \in X\} \cup \{(x,y) | x \neq y\} = Y \cup Z.$$

Both Y and Z are G-stable, and so $\chi_{X \times X} = \chi_Y + \chi_Z$. Since $|X| > 1$, both of these sets are nonempty.

The action of G on Y is transitive because the action on X is transitive. By assumption, the action of G on Z is also transitive.

Therefore,

$$
\begin{aligned}
\langle \chi_X, \chi_X \rangle &= \langle \chi_Y + \chi_Z, \mathbf{1} \rangle \\
&= \langle \chi_Y, \mathbf{1} \rangle + \langle \chi_Z, \mathbf{1} \rangle \\
&= 1 + 1 = 2.
\end{aligned}
$$