18.704 Problem Set 4 Solutions

1. S_3 has the following 6 subgroups:

$$\{e\}, \{e, (12)\}, \{e, (13)\}, \{e, (23)\}, \{e, (123), (132)\}, S_3.$$

Here are their character tables. (The three groups of size 2 all have the same character tables.) Write $H = \{e, (12)\}$ and $K = \{e, (123), (132)\}$.

$\{e\}$	e
$\mathbf{1}$	1

H	e	(12)
$\mathbf{1}$	1	1
ϵ	1	-1

K	e	(123)	(132)
$\mathbf{1}$	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

(Here $\omega = e^{\pi i/3}$.)

S_3	e	(12)	(123)
$\mathbf{1}$	1	1	1
sgn	1	-1	1
T	2	0	-1

Here are the characters of the induced representations to S_3.

S_3	e	(12)	(123)
$\mathrm{Ind}_{\{e\}}^{S_3}(\mathbf{1})$	6	0	0
$\mathrm{Ind}_{H}^{S_3}(\mathbf{1})$	3	1	0
$\mathrm{Ind}_{H}^{S_3}(\epsilon)$	3	-1	0
$\mathrm{Ind}_{K}^{S_3}(\mathbf{1})$	2	0	2
$\mathrm{Ind}_{K}^{S_3}(\chi_2)$	2	0	-1
$\mathrm{Ind}_{K}^{S_3}(\chi_3)$	2	0	-1

Writing these characters as sums of irreducible characters, we get the following.

$$\begin{aligned}
\mathrm{Ind}_{\{e\}}^{S_3} &= \mathbf{1} + \mathrm{sgn} + 2T \\
\mathrm{Ind}_{H}^{S_3}(\mathbf{1}) &= \mathbf{1} + T \\
\mathrm{Ind}_{H}^{S_3}(\epsilon) &= \mathrm{sgn} + T \\
\mathrm{Ind}_{K}^{S_3}(\mathbf{1}) &= \mathbf{1} + \mathrm{sgn} \\
\mathrm{Ind}_{K}^{S_3}(\chi_2) &= T \\
\mathrm{Ind}_{K}^{S_3}(\chi_3) &= T
\end{aligned}$$

2. If $f_1, f_2 \in W, \alpha \in \mathbb{C}$, then we have the following.

$$\begin{aligned}(\alpha f_1 + f_2)(hx) &= \alpha f_1(hx) + f_2(hx) \\ &= \alpha\rho(h)f_1(x) + \rho(h)f_2(x) \\ &= \rho(h)(\alpha f_1(x) + f_2(x)) \\ &= \rho(h)(\alpha f_1 + f_2)(x).\end{aligned}$$

   Therefore, $\alpha f_1 + f_2 \in W$, as desired.

   If $g \in G, f \in W$, then we have the following.

$$\begin{aligned}(g \cdot f)(hx) &= f(hxg^{-1}) \\ &= \rho(h)f(xg^{-1}) \\ &= \rho(h)(g \cdot f)(x).\end{aligned}$$

   Therefore, $g \cdot f \in W$.

   To show that this gives a representation of G on W, we need to show the identities $e \cdot f = f$ and $g \cdot (g' \cdot f) = (gg') \cdot f$. Since

$$(e \cdot f)(x) = f(xe^{-1}) = f(x),$$

   we have $e \cdot f = f$. Also, we have the following.

$$\begin{aligned}(g \cdot (g' \cdot f))(x) &= (g' \cdot f)(xg^{-1}) \\ &= f(xg^{-1}(g')^{-1}) \\ &= f(x(g'g)^{-1}) \\ &= (g'g) \cdot f(x).\end{aligned}$$

   At this point, we realize that I made a **MISTAKE** when I assigned this pset; this doesn't give an action of G on W because it is backwards! Instead, we should have defined the group action via

$$(g \cdot f)(x) = f(xg)$$

   instead!

3. Here is the starting point for our character table, whose rows are the characters of S_5 acting on tabloids.

	$(1,1,1,1,1)$ (1)	$(2,1,1,1)$ (10)	$(2,2,1)$ (15)	$(3,1,1)$ (20)	$(3,2)$ (20)	$(4,1)$ (30)	(5) (24)
(5)	1	1	1	1	1	1	1
$(4,1)$	5	3	1	2	0	1	0
$(3,2)$	10	4	2	1	1	0	0
$(3,1,1)$	20	6	0	2	0	0	0
$(2,2,1)$	30	6	2	0	0	0	0
$(2,1,1,1)$	60	6	0	0	0	0	0
$(1,1,1,1,1)$	120	0	0	0	0	0	0

First, we kill copies of the trivial representation (5), which occur once in each lower row.

	$(1,1,1,1,1)$ (1)	$(2,1,1,1)$ (10)	$(2,2,1)$ (15)	$(3,1,1)$ (20)	$(3,2)$ (20)	$(4,1)$ (30)	(5) (24)
(5)	1	1	1	1	1	1	1
$(4,1)$	4	2	0	1	−1	0	−1
$(3,2)$	9	3	1	0	0	−1	−1
$(3,1,1)$	19	5	−1	1	−1	−1	−1
$(2,2,1)$	29	5	1	−1	−1	−1	−1
$(2,1,1,1)$	59	5	−1	−1	−1	−1	−1
$(1,1,1,1,1)$	119	−1	−1	−1	−1	−1	−1

Then we kill off copies of the now-irreducible representation $(4,1)$, which occurs once in $(3,2)$, twice in $(3,1,1)$, twice in $(2,2,1)$, three times in $(2,1,1,1)$, and four times in $(1,1,1,1,1)$.

	$\yng(1,1,1,1,1)$ (1)	$\yng(2,1,1,1)$ (10)	$\yng(2,2,1)$ (15)	$\yng(3,1,1)$ (20)	$\yng(3,2)$ (20)	$\yng(4,1)$ (30)	$\yng(5)$ (24)
$\yng(5)$	1	1	1	1	1	1	1
$\yng(4,1)$	4	2	0	1	−1	0	−1
$\yng(3,2)$	5	1	1	−1	1	−1	0
$\yng(3,1,1)$	11	1	−1	−1	1	−1	1
$\yng(2,2,1)$	21	1	1	−3	1	−1	1
$\yng(2,1,1,1)$	47	−1	−1	−4	2	−1	2
$\yng(1,1,1,1,1)$	103	−9	−1	−5	3	−1	3

Next, the irreducible $\yng(3,2)$ occurs once in $\yng(3,1,1)$, twice in $\yng(2,2,1)$, three times in $\yng(2,1,1,1)$, and five times in $\yng(1,1,1,1,1)$.

	$\yng(1,1,1,1,1)$ (1)	$\yng(2,1,1,1)$ (10)	$\yng(2,2,1)$ (15)	$\yng(3,1,1)$ (20)	$\yng(3,2)$ (20)	$\yng(4,1)$ (30)	$\yng(5)$ (24)
$\yng(5)$	1	1	1	1	1	1	1
$\yng(4,1)$	4	2	0	1	−1	0	−1
$\yng(3,2)$	5	1	1	−1	1	−1	0
$\yng(3,1,1)$	6	0	−2	0	0	0	1
$\yng(2,2,1)$	11	−1	−1	−1	−1	1	1
$\yng(2,1,1,1)$	32	−4	−4	−1	−1	2	2
$\yng(1,1,1,1,1)$	78	−14	−6	0	−2	4	3

The irreducible $\yng(3,1,1)$ occurs once in $\yng(2,2,1)$, three times in $\yng(2,1,1,1)$, and six times in $\yng(1,1,1,1,1)$.

	$(1,1,1,1,1)$ (1)	$(2,1,1,1)$ (10)	$(2,2,1)$ (15)	$(3,1,1)$ (20)	$(3,2)$ (20)	$(4,1)$ (30)	(5) (24)
(5)	1	1	1	1	1	1	1
$(4,1)$	4	2	0	1	−1	0	−1
$(3,2)$	5	1	1	−1	1	−1	0
$(3,1,1)$	6	0	−2	0	0	0	1
$(2,2,1)$	5	−1	1	−1	−1	1	0
$(2,1,1,1)$	14	−4	2	−1	−1	2	−1
$(1,1,1,1,1)$	42	−14	6	0	−2	4	−3

The irreducible $(2,2,1)$ occurs twice in $(2,1,1,1)$ and five times in $(1,1,1,1,1)$.

	$(1,1,1,1,1)$ (1)	$(2,1,1,1)$ (10)	$(2,2,1)$ (15)	$(3,1,1)$ (20)	$(3,2)$ (20)	$(4,1)$ (30)	(5) (24)
(5)	1	1	1	1	1	1	1
$(4,1)$	4	2	0	1	−1	0	−1
$(3,2)$	5	1	1	−1	1	−1	0
$(3,1,1)$	6	0	−2	0	0	0	1
$(2,2,1)$	5	−1	1	−1	−1	1	0
$(2,1,1,1)$	4	−2	0	1	1	0	−1
$(1,1,1,1,1)$	17	−9	1	5	3	−1	−3

Finally, the representation $(2,1,1,1)$ occurs four times in $(1,1,1,1,1)$.

	(1,1,1,1,1) (1)	(2,1,1,1) (10)	(2,2,1) (15)	(3,1,1) (20)	(3,2) (20)	(4,1) (30)	(5) (24)
(5)	1	1	1	1	1	1	1
(4,1)	4	2	0	1	−1	0	−1
(3,2)	5	1	1	−1	1	−1	0
(3,1,1)	6	0	−2	0	0	0	1
(2,2,1)	5	−1	1	−1	−1	1	0
(2,1,1,1)	4	−2	0	1	1	0	−1
(1,1,1,1,1)	1	−1	1	1	−1	−1	1

(And the chances of us getting the sign representation down at the bottom by *accident* are pretty low, so I think the arithmetic worked out.)