18.906 Problem Set 2

Due Wednesday, February 21 in class

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1. Quickies.

   - Show that the Hurewicz map $\pi_n(X, x) \to \tilde{H}_n(X) \cong H_n(X, x)$ is a group homomorphism for $n > 0$. (Hint: Consider the effect on homology of the pinch map $S^n \to S^n \vee S^n$.)
   - Prove the lemma mentioned in class: If A is a set with two binary operations $*, \circ$ that have the same identity element $e \in A$ and satisfy an interchange law

     $$(a * b) \circ (c * d) = (a \circ c) * (b \circ d),$$

     show that $* = \circ$ and that $a * b = b * a$ for all $a, b \in A$.

2. Suppose that a group G acts properly discontinuously on a space Y on the left, and suppose that A, B are discrete sets with right actions of G. We can construct a space

   $$A \times_G Y = A \times Y / \{(ag, y) \sim (a, gy) \mid g \in G\},$$

   and similarly for B. Suppose $f : A \to B$ satisfies $f(ag) = f(a)g$ for all $a \in A, g \in G$. Show that there is an induced covering map $A \times_G Y \to B \times_G Y$. Give an explicit description of the preimage of a point of $B \times_G Y$.

3. A *fiber bundle with fiber* F is a map $p : E \to B$ of spaces such that, for every $b \in B$, there exists a neighborhood U of b and a homeomorphism $\phi : U \times F \to p^{-1}U$ such that $p\phi(u, f) = u$ for all $(u, f) \in U \times F$.

   Show that fiber bundles have a disc lifting property, as follows. Let $D^{n-1} \subset D^n = [0, 1]^n$ be the "cap", consisting of the union of all faces but one. Suppose that we have a map $g : D^n \to B$ which has a chosen lift on the cap $\tilde{g} : D^{n-1} \to E$. Show that there exists an extension (not necessarily unique) to a lift $\tilde{g} : D^n \to E$. (Hint: Subdivide, and use induction on n.)

4. In this exercise, we will show some "exactness" properties of the long exact sequence in homotopy in low degrees. Recall that $\pi_1(X, A, x)$ is acted on by the group $\pi_1(X, x)$ by path composition: if γ is a loop based at x and λ is a path starting at x and ending in A, we can form the composition $\gamma\lambda$. In particular, the image of γ in $\pi_1(X, A, x)$ is γ times the trivial element.

   Show that two elements γ, γ' of $\pi_1(X, x)$ have the same image in $\pi_1(X, A, x)$ if and only if $\gamma = \gamma'\alpha$ for some α in the image of $\pi_1(A, x)$.

   Show that that two elements λ, λ' in $\pi_1(X, A, x)$ have the same image in $\pi_0(A)$ if and only if $\lambda = \gamma\lambda'$ for some $\gamma \in \pi_1(X, x)$.

   Show that an element of $\pi_0(A)$ maps to the component of x in $\pi_0(X)$ if and only if it lifts to $\pi_1(X, A, x)$.