18.906 Problem Set 3

Due Wednesday, February 28 in class

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1. Quickies.
   - Let $X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Show that the inclusion $\{0\} \hookrightarrow X$ is not a cofibration.
   - Show that if $A \to B$ is a cofibration of compactly generated Hausdorff spaces, so is $A \times [0,1] \to B \times [0,1]$.

2. Suppose X is a space and $f, g : S^n \to X$ are *freely* homotopic maps (meaning that we do not require the homotopy to preserve the basepoint). Let Cf and Cg be the mapping cones of f and g, formed by attaching $(n+1)$-cells using the attaching maps f and g.

   Explicitly show that we can construct maps $\phi : Cf \to Cg$ and $\psi : Cg \to Cf$ such that $\phi|_X = \psi|_X = id_X$, together with homotopies H from $\phi \circ \psi$ to id_{Cg} and H' from $\psi \circ \phi$ to id_{Cf} which restrict to the constant homotopy on X.

3. Suppose that X is a space with basepoint x, and we have two based maps $f, g : S^n \to X$. Show that f and g are freely homotopic if and only if there exists an element $\gamma \in \pi_1(X, x)$ such that the action of γ on $[f] \in \pi_n(X, x)$ gives $[g] \in \pi_n(X, x)$.

4. In this exercise we will construct a family of fibrations that don't look much like fiber bundles.

   Suppose that $B = U \cup V$ for open subsets U and V. Define a space

   $$E = \{(b, t) \in B \times [0,1] \mid t = 0 \text{ if } b \notin U,\ t = 1 \text{ if } b \notin V\}.$$

   E is formed by gluing $[0,1]$ times $U \cap V$ to U and V at the ends.

   Show that the obvious projection map $p : E \to B$ is an *acyclic Serre fibration*: if we have a map $f : D^n \to B$ and a map $\tilde{f} : \partial D^n \to E$ such that $p\tilde{f} = f|_{\partial D^n}$, show that there exists an extension $\tilde{f} : D^n \to E$ such that $p\tilde{f} = f$. (Hint: You might need a theorem from elementary point-set topology.)

   Show that an acyclic Serre fibration p automatically induces isomorphisms $p_* : \pi_n(E, e) \to \pi_n(B, p(e))$ for all basepoints $e \in E$, $n > 0$, and an isomorphism $\pi_0(E) \to \pi_0(B)$.