18.906 Problem Set 4

Due Wednesday, March 7 in class

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1. Show that if K is a finite CW-complex with cells of dimension less than or equal to d and $Y \to X$ is a map of spaces, then the induced map

$$[K, Y] \to [K, X]$$

is surjective if the map $Y \to X$ is d-connected and an isomorphism if the map $Y \to X$ is $(d+1)$-connected. (Recall that a map is k-connected if it is an isomorphism on homotopy in degree less than k and an epimorphism in degree k.)

2. Let G be a compact Lie group with closed subgroup H (which is also a Lie group). Show that the map $G \to G/H$ is a fibration with fiber H.

   If $O(n)$ is the Lie group of $n \times n$ orthogonal matrices and $U(n)$ is the Lie group of $n \times n$ unitary matrices, use this to show that the map $O(n) \to O(n+1)$ is n-connected and the map $U(n) \to U(n+1)$ is $2n+1$-connected.

3. Prove the following version of the *Whitehead theorem for chain complexes*: If $i : C_* \subset D_*$ is an inclusion of chain complexes such that

   - $C_n = D_n = 0$ for $n < 0$,
   - D_n/C_n is free (or projective, if you prefer) for all n, and
   - the induced map $H_n(C_*) \to H_n(D_*)$ is an isomorphism for all n.

   Show that there is a map $r : D_* \to C_*$ such that ir and ri are chain homotopic to identity maps.[1]

4. Suppose $p : E \to B$ is a Hurewicz fibration and $F = p^{-1}(*)$ be the preimage of the basepoint. Let γ be any loop in B. Show that there exists a map $G : F \times [0,1] \to E$ such that $G(f,0) = f$ and $pG(f,t) = \gamma(t)$ for all f and t. Show that if γ and γ' have associated maps G and G', and $h : [0,1] \times [0,1]$ is a homotopy from γ to γ', there exists a map $H : F \times [0,1] \times [0,1] \to E$ such that $H(f,t,0) = G(f,t)$, $H(f,t,1) = G'(f,t)$, and $pH(f,t,s) = h(t,s)$.

   Define $L(\gamma) : F \to F$ to be the restriction of G to $F \times \{1\}$. Show that paths homotopic to γ or different choices of lift give rise to freely homotopic maps $L(\gamma)$. Show that $L(\delta) \circ L(\gamma)$ is freely homotopic to $L(\gamma\delta)$ for all loops γ and δ.

   Conclude that there is a (right) action of $\pi_1(B,*)$ on $H_n(F)$ for all n. Why is there not necessarily an action of $\pi_1(B,*)$ on $\pi_n(F,*)$? (Note:

[1] There is a lot of similarity between the study of chain complexes and spaces. Both are *model categories*, which have notions of cofibrations, fibrations, and equivalences that satisfy various axioms, allowing many of the standard proofs to carry through.

This action exists for Serre fibrations as well, but we restrict to Hurewicz fibrations for simplicity.)