18.906 Problem Set 6

Due Wednesday, March 21 in class

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1. Compute the rational cohomology of ΩS^n as a ring, where n is even.

2. Suppose that X is 1-connected. Use the Serre spectral sequence to show the (co)homology groups of X are finitely generated if and only if the (co)homology groups of the loop space ΩX are. Same question for finite instead of finitely generated.

3. As in a previous assignment, the group S^1 acts on $S^{2n-1} \subset \mathbb{C}^n$ via

$$\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n).$$

   There is an induced action on the union $S^\infty = \cup S^{2n-1}$.

   Let $C_n \subset S^1$ be the cyclic subgroup of order n. The quotient space S^∞/C_n is a $K(\mathbb{Z}/n, 1)$.

   The group S^1/C_n acts on S^∞/C_n with quotient $\mathbb{CP}^\infty$, and there is a resulting fibration sequence $S^1 \to K(\mathbb{Z}/n, 1) \to \mathbb{CP}^\infty$. Use the Serre spectral sequence to show that the cohomology groups of $K(\mathbb{Z}/n, 1)$ are finite except in dimension 0. (Hint: Use the Hurewicz isomorphism to find the first nonvanishing homology and cohomology groups, and use that to determine differentials.)

4. Using the previous exercises, show that $H^p(K(\mathbb{Z}, m))$ is finitely generated for all $p, m > 0$. Show that $H^p(K(\mathbb{Z}/n, m))$ is finite for all $p, n, m > 0$.