18.906 Problem Set 8

Due Wednesday, April 11 in class

In this problem set, we'll cover a lot of steps in constructing (most) Pontriagin classes. Recall that $SO(n)$ is the group of $n \times n$ orthogonal matrices with determinant $+1$, and $SO(n)$ is connected.

1. Use Question 4 on problem set 6 to show that $\tilde{H}_p(K(A, m); \mathbb{Q}) = 0$ for all finite abelian groups A and all $m > 0$. Use the Serre spectral sequence and Postnikov towers to conclude that if X is a 1-connected space whose homotopy groups are all finite, then $H_*(X; \mathbb{Q}) = 0$. (You can use without proof the fact that cohomology with rational coefficients is dual to homology with rational coefficients, so that one vanishes if and only if the other does.)

   Use this to prove the *rational Hurewicz theorem*: if X is 1-connected and the rationalization $\pi_k(X, x) \otimes \mathbb{Q}$ is 0 for $k < n$, then $\pi_n(X, x) \otimes \mathbb{Q} \cong H_n(X; \mathbb{Q})$. (Use Postnikov towers again.)

2. Let $F(2, n) = O(n)/O(n-2) = SO(n)/SO(n-2)$. Explain why $F(2, n)$ can be identified with the space of (ordered) pairs of orthogonal vectors in $\mathbb{R}^n$. Show that there is a fibration $F(2, n) \to S^{n-1}$ with fiber S^{n-2}. (Here $n \geq 3$.)

   Use disc lifting to explicitly show that the connecting homomorphism

   $$\pi_{n-1}(S^{n-1}) \to \pi_{n-2}(S^{n-2})$$

   from the long exact sequence of this fibration is zero if n is even, and multiplication by 2 if n is odd.

   Conclude that

   $$H_{n-2}(F(2, n); \mathbb{Z}/2) = \mathbb{Z}/2$$

   for all n, and

   $$H_{n-2}(F(2, n); \mathbb{Q}) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{Q} & \text{if } n \text{ is even.} \end{cases}$$

3. Use problem 2 to show that if n is odd, then $\pi_{n-2}(F(2, n)) \otimes \mathbb{Q} = 0$. Use problem 1 to conclude that $H_{n-1}(F(2, n); \mathbb{Q}) = 0$, and then use this knowledge together with the Serre spectral sequence to show that if n is odd,

   $$H^k(F(2, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 0, 2n-3 \\ 0 & \text{otherwise.} \end{cases}$$

   ($n = 3$ is a special case in this problem.)

4. Use the fibration sequence $SO(n-2) \to SO(n) \to F(2,n)$ to inductively show that for $n \geq 3$ odd,

$$H^*(SO(n);\mathbb{Q}) \cong \mathbb{Q}[z_3, z_7, z_{11}, \cdots, z_{2n-3}]/(z_i^2),$$

an exterior algebra on classes in degrees $4k-1$. (For $n=3$ you can assume that $\pi_1(F(2,3))$ acts trivially on $H_*(SO(1))$ - it does - so that the Serre spectral sequence exists.)