

Derivatives

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Definition: The *derivative* with respect to x of a function $y = f(x)$ is **the function** $f'(x)$ defined by

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \left\{ \text{also denoted by } \frac{d}{dx} f(x) = \frac{df}{dx} = \frac{dy}{dx} = y' = \dots \right\}$$

We say that $f(x)$ is *differentiable* at $x = c$ if the above limit exists at $x = c$.

Remarks

1. **Rate of Change:** The derivative $f'(x)$ of a function $f(x)$ is the *instantaneous* rate of change of $f(x)$.
2. **Slope of Graph:** The derivative $f'(x)$ of a function $f(x)$ is the *slope* of the curve $y = f(x)$.

Differentiation Rules

Let $f(x)$ and $g(x)$ be any two differentiable functions. Let c be a *constant* and a_0, a_1, \dots, a_n be real numbers. Then

1. **Constant Rule:** $\frac{d}{dx} [c] = 0$
2. **Power Rule:** For any non-zero real number n , $\frac{d}{dx} [x^n] = n \cdot x^{n-1}$
3. **Power Rule (Special case):** $\frac{d}{dx} [x] = 1$
4. **Constant Multiple Rule:** $\frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} [f(x)]$
5. **Sum Rule:** $\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$
6. **Difference Rule:** $\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} [f(x)] - \frac{d}{dx} [g(x)]$
7. **Polynomial Rule (Combining all the above rules):** For any positive integer n ,
 $\frac{d}{dx} [a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0] = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + 2 a_2 x + a_1$
8. **Product Rule:** $\frac{d}{dx} [f(x) \cdot g(x)] = \frac{d}{dx} [f(x)] \cdot g(x) + f(x) \frac{d}{dx} [g(x)]$
9. **Quotient Rule:** $\frac{d}{dx} [f(x)/g(x)] = \frac{\frac{d}{dx} [f(x)] g(x) - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$ provided $g(x) \neq 0$
10. **Chain Rule:** For any two differentiable functions $y = f(u)$ and $u = g(x)$,
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ that is, $\frac{d}{dx} [f \circ g(x)] = \frac{d}{du} [f(u)] \Big|_{u=g(x)} \cdot \frac{d}{dx} [g(x)] = f'(g(x)) \cdot g'(x)$
11. **General Power Rule:** For any non-zero real number n , $\frac{d}{dx} [f(x)]^n = n \cdot [f(x)]^{n-1} \cdot \frac{d}{dx} [f(x)]$

Higher Order Derivatives

1. The second derivative of a function $y = f(x)$ is the derivative of its derivative. It is denoted by
 $\frac{d}{dx} \left[\frac{d}{dx} [f(x)] \right] = (f')' = f'' = \frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d^2 y}{dx^2}$
2. For any positive integer n , the n -th derivative of a function $y = f(x)$ is the derivative of its $(n-1)$ -th derivative, that is, the function obtained by differentiating $f(x)$ successively n times.
 $\frac{d}{dx} [f^{(n-1)}(x)] = (f^{(n-1)})' = f^{(n)} = \frac{d}{dx} \left[\frac{d^{(n-1)} y}{dx^{(n-1)}} \right] = \frac{d^n y}{dx^n}$