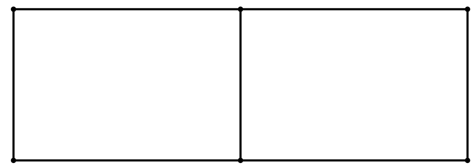
Here is the key to the fourth exam. This exam covered chapters 3 and 7 from the book. Be sure you understand how to do each problem since you will see similar problems on the final exam.

1. (9 points) A farmer wishes to use 72 feet of fencing to surround two identical adjacent rectangular plots. If the farmer chooses the dimensions of the plots that would maximize the enclosed area, what would the area of each plot.



   a. 216 sq feet

   b. 108 sq feet

   c. 72 sq feet

   d. 144 sq feet

   e. None of the above

Solution:

Ans: b. 108 sq feet

2. (9 points) A manufacturer can produce sunglasses at a cost of \$2 apiece and estimates that if they are sold for x dollars apiece, consumers will buy $16(12 - x)$ sunglasses a day. At what price should the manufacturer sell the sunglasses to maximize his profit.

   a. 5 dollars
   b. 6 dollars
   c. 7 dollars
   d. 8 dollars
   e. None of the above

Solution:

Ans: c. 7 dollars

3. (9 points) A window is in the form of an equilateral triangle surmounted on a rectangle. The rectangle transmits twice as much light as does the triangle. If the entire window has a perimeter of 20 feet, find the width of the window that will admit the most light. (Note: The area of an equilateral triangle with side s is $\frac{\sqrt{3}}{4}s^2$.)

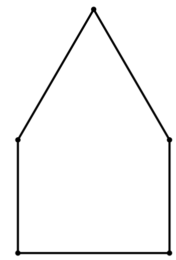
   a. 3.9 feet

   b. 4.0 feet

   c. 4.1 feet

   d. 4.2 feet

   e. None of the above



Solution:

Ans: a. 3.9 feet

4. (9 points) What is the maximum volume of a cylindrical can with no top that can be made from 27π square inches of metal?

   a. 27 cubic inches
   b. 9 cubic inches
   c. 3 cubic inches
   d. 1 cubic inches
   e. None of the above

Solution:

Ans: e. None of the above

5. (9 points) Find the maximum and/or minimum values of the function $f(x, y) = x^2 + 2y^2 - xy$ subject to the constraint $2x + y = 22$.

   a. 66
   b. 77
   c. 88
   d. 99
   e. None of the above

Solution:

Ans: b. 77

6. (20 points) At a certain factory, the daily output is approximately $40K^{1/3}L^{1/2}$ units, where K denotes the capital investment measured in units of \$1,000 and L denotes the size of labor force measured in worker-hours. Suppose the current capital investment is \$125,000 and that 900 worker-hours of labor are used each day. Determine the effect that an additional capital investment of \$1,000 will have on the daily output if the size of the labor force is not changed.

Solution: The daily output is

$$O(K, L) = 40K^{1/3}L^{1/2}\,.$$

Therefore,

$$O(125, 900) = 40 \cdot 125^{1/3} \cdot 900^{1/2} = 6000$$

and

$$O(126, 900) = 40 \cdot 126^{1/3} \cdot 900^{1/2} \approx 6016\,.$$

Therefore the daily output will increase by *approximately* 16 units.

7. (20 points) Compute the first-order partial derivatives f_x and f_y and the second-order partial derivatives f_{xx}, f_{xy}, f_{yy} and f_{yx} for the function $f(x,y) = xye^{xy}$.

Solution: Given $f(x,y) = xye^{xy}$. Therefore

$$
\begin{aligned}
f_x(x,y) &= y\left(e^{xy} + xe^{xy} \cdot y\right) \\
&= ye^{xy}\left(1 + xy\right)
\end{aligned}
$$

$$
\begin{aligned}
f_y(x,y) &= x\left(e^{xy} + ye^{xy} \cdot x\right) \\
&= xe^{xy}\left(1 + xy\right)
\end{aligned}
$$

$$
\begin{aligned}
f_{xx}(x,y) &= y\left(e^{xy} \cdot y\left(1 + xy\right) + e^{xy}\left(0 + y\right)\right) \\
&= y^2e^{xy}\left(2 + xy\right)
\end{aligned}
$$

$$
\begin{aligned}
f_{yy}(x,y) &= x\left(e^{xy} \cdot x\left(1 + xy\right) + e^{xy}\left(0 + x\right)\right) \\
&= x^2e^{xy}\left(2 + xy\right)
\end{aligned}
$$

$$
\begin{aligned}
f_{xy}(x,y) &= 1 \cdot e^{xy}\left(1 + xy\right) + ye^{xy} \cdot x\left(1 + xy\right) + ye^{xy}\left(0 + x\right) \\
&= e^{xy}\left(1 + xy + xy + (xy)^2 + yx\right) \\
&= e^{xy}\left(1 + 3xy + (xy)^2\right) && \Big(= f_{yx}(x,y)\Big) \\
& && \quad || \\
f_{yx}(x,y) &= 1 \cdot e^{xy}\left(1 + xy\right) + xe^{xy} \cdot y\left(1 + xy\right) + xe^{xy}\left(0 + y\right) && \quad || \\
&= e^{xy}\left(1 + xy + xy + (xy)^2 + xy\right) && \quad || \\
&= e^{xy}\left(1 + 3xy + (xy)^2\right) && \Big(= f_{xy}(x,y)\Big)
\end{aligned}
$$

8. (20 points) Compute the first-order partial derivatives f_x and f_y and the second-order partial derivatives f_{xx}, f_{xy}, f_{yy} and f_{yx} for the function $f(x, y) = x \ln y$.

Solution: Given $f(x, y) = x \ln y$. Therefore

$$f_x(x, y) = \ln y$$

$$\begin{aligned} f_y(x, y) &= x \cdot \frac{1}{y} \\ &= \frac{x}{y} \end{aligned}$$

$$f_{xx}(x, y) = 0$$

$$\begin{aligned} f_{yy}(x, y) &= x \cdot \frac{-1}{y^2} \\ &= -\frac{x}{y^2} \end{aligned}$$

$$f_{xy}(x, y) = \frac{1}{y} \qquad \Big(= f_{yx}(x, y)\Big)$$

$$||$$

$$f_{yx}(x, y) = \frac{1}{y} \qquad \Big(= f_{xy}(x, y)\Big)$$

9. (20 points) Find the critical points and classify each as a relative maximum, relative minimum or a saddle point for the the function $f(x,y) = x^2 + y^3 + 6xy - 7x - 6y$.

Solution: Given $f(x,y) = x^2 + y^3 + 6xy - 7x - 6y$. Therefore

$$\begin{aligned}
f_x(x,y) &= 2x + 0 + 6y - 7 - 0 = 2x + 6y - 7 \\
f_y(x,y) &= 0 + 3y^2 + 6x - 0 - 6 = 3y^2 + 6x - 6 \\
f_{xx}(x,y) &= 2 + 0 - 0 = 2 \\
f_{xy}(x,y) &= 0 + 6 - 0 = 6 && \Big(= f_{yx}(x,y)\Big) \\
f_{yy}(x,y) &= 3 \cdot 2y^1 + 0 - 0 = 6y && || \\
f_{yx}(x,y) &= 0 + 6 - 0 = 6 && \Big(= f_{xy}(x,y)\Big) \\
D(x,y) &= f_{xx} \cdot f_{yy} - (f_{xy})^2 \\
&= 2 \cdot 6y - 6^2 = 12y - 36
\end{aligned}$$

Now, $f_x(x,y) = 0$ if $2x + 6y - 7 = 0$. That is when

$$x = \frac{-6y + 7}{2} \tag{1}$$

Simultaneously, $f_y(x,y) = 0$ if $3y^2 + 6x - 6 = 0$. That is when

$$\begin{aligned}
& 3y^2 + 6\,\frac{-6y+7}{2} - 6 = 0 && \text{from (1)} \\
\text{or} \quad & 3y^2 - 18y + 15 = 0 \\
\text{or} \quad & 3(y-1)(y-5) = 0 \\
\text{or} \quad & y = 1 \quad \text{or} \quad y = 5 && (2) \\
\text{and} \quad & x = \frac{-6+7}{2} = \frac{1}{2} \quad \text{or} \quad x = \frac{-6 \cdot 5 + 7}{2} = -\frac{23}{2} && \text{from (1)} \quad (3)
\end{aligned}$$

So, the critical points are $(\frac{1}{2}, 1)$ and $(-\frac{23}{2}, 5)$. Further,

$$\begin{aligned}
D\Big(\frac{1}{2}, 1\Big) &= 12 \cdot 1 - 36 = -24 < 0 \\
D\Big(-\frac{23}{2}, 5\Big) &= 12 \cdot 5 - 36 = 24 > 0 \\
\text{and} \quad f_{xx}\Big(-\frac{23}{2}, 5\Big) &= 2 > 0
\end{aligned}$$

Therefore, $(\frac{1}{2}, 1, -\frac{21}{4})$ is a *saddle point* and $(-\frac{23}{2}, 5, -37)$ is a *local minimum*.

PS: **Please check:** $f(\frac{1}{2}, 1) = -\frac{21}{4}$ **and** $f(-\frac{23}{2}, 5) = -37$.