LECTURE 5: GENERALIZATIONS OF OPERADS

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1. Generalizations of operads

1.1. PROP's and operads. We have approached operads from the point of view of PROP's, tensor categories whose set of objects is the set of nonnegative integers with the sum as the tensor product, see Lecture 1. If we take the sets $\mathrm{Mor}(n, 1)$ of morphism for $n \geq 0$ in a PROP, we will obtain an operad.

There is a converse construction of a minimal PROP, freely generated by an operad. Suppose we have an operad $\mathcal{O}$ of sets. Define a PROP by defining the set of morphisms as

$$\mathrm{Mor}(m, n) = \bigcup_{m_1 + \dots + m_n = m} \mathcal{O}(m_1) \otimes \dots \otimes \mathcal{O}(m_n) \times_G S_m,$$

where the summation runs over all sequences $m_1, \dots, m_n \geq 0$ summing up to m, $G = S_{m_1} \times \dots \times S_{m_n}$, naturally embedded in S_m. The product with S_m over G is meant to allow all possible ways of labeling the inputs and provide an action of S_m on the labels.

Notice that starting from an operad, going to the corresponding PROP and then getting back to an operad, returns the original operad. On the other hand, if you start from a PROP, construct the corresponding operad and then the corresponding PROP, then the new PROP will be in general different from the old one. Therefore, the theory of PROP's is richer than that of operads, but one can think of an operad as a more basic object.

1.2. Modular operads.

Definition 1.1. A *modular operad* is a collection of spaces $\mathcal{O}(n)$, $n \geq 0$, along with an S_n-action on each $\mathcal{O}(n)$ and two types of compositions:

$$\circ_{ij} : \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m + n - 2), \qquad 1 \leq i \leq m, 1 \leq j \leq n,$$
$$\circ_{ij} : \mathcal{O}(n) \to \mathcal{O}(n - 2), \qquad 1 \leq i < j \leq n,$$

satisfying natural associativity and equivariance properties.

Modular operads were introduced by Getzler and Kapranov in [GK96] as more symmetric objects than PROP's. The main example they had in mind was the moduli spaces (whence the term "modular") $\mathcal{M}(n) = \bigcup_{g \geq 0} \overline{\mathcal{M}}_{g,n}$ of stable complex compact algebraic curves of an arbitrary genus g with n punctures. The $\circ_{ij}$ operations are the operations of joining two punctures on two different curves or on a single curve to form a double point. The more familiar to us space $\mathcal{P}(n)$ of smooth complex compact algebraic curves (or Riemann surfaces) of an arbitrary genus with n holomorphic holes is another example of a modular operad.

Date: September 30, 1997.

Getzler and Kapranov used a different definition of a modular operad, which was particularly suitable to handle the moduli spaces of stable curves. The problem with the moduli space of stable curves with punctures is essentially that not all combinations of genus g and the number of punctures n are allowed: the Euler characteristic $2-2g_c-n_c$ of each irreducible component c of a stable curve must be negative. Just for completeness, we would like to define modular operads in their sense, as well.

Definition 1.2. A *stable* (*i.e., Getzler-Kapranov*) *modular operad* is a collection of spaces $\mathcal{O}(g,n)$, $g,n \geq 0$, $2-2g-n<0$, along with an S_n-action on each $\mathcal{O}(g,n)$ and two types of compositions:

$$\circ_{ij} : \mathcal{O}(g_1,m) \otimes \mathcal{O}(g_2,n) \to \mathcal{O}(g_1+g_2, m+n-2), \qquad 1 \leq i \leq m, 1 \leq j \leq n,$$
$$\circ_{ij} : \mathcal{O}(g,n) \to \mathcal{O}(g+1, n-2), \qquad 1 \leq i < j \leq n,$$

satisfying natural associativity and equivariance properties.

Exercise 1. Construct functors between operads, modular operads, and PROP's and study their relationship.

2. Operads generalizing those of Riemann surfaces

The previous section dealt with generalizations of operads, whereas here we would like to consider (modular) operads generalizing those of Riemann surfaces and yet relevant to 2d QFT's. First of all, those relevant to super CFT's.

2.1. **The operad of super Riemann surfaces.** A *super Riemann surface or a SUSY curve* is a complex supermanifold of dimension $1|1$ with a subbundle $S \subset \mathcal{T}$ of dimension $0|1$ in the holomorphic tangent bundle $\mathcal{T}$, satisfying the following nonintegrability condition: the morphism

$$S \otimes S \to \mathcal{T}/S,$$
$$X_1 \otimes X_2 \mapsto [X_1, X_2] \mod S,$$

$[X_1, X_2]$ being the super commutator of vector fields, is an isomorphism. Usually, it makes more sense to consider families $X \to B$ of super Riemann surfaces — then one just replaces the holomorphic tangent bundle with the relative one.

A trivial example of a super Riemann surface is the standard unit disk $D^{1|1} = \{(z,\zeta) \in \mathbb{C}^{1|1} \mid |z| < 1\}$ with the subbundle S spanned by the odd vector field $X = \partial/\partial\zeta + \zeta\partial/\partial z$ (Note that $[X,X] = 2\partial/\partial z$). Thus, it is clear what a super Riemann surface with a holomorphic hole would be. The moduli spaces of such form a PROP (as well as an operad and a modular operad), and an algebra over such PROP (operad) would be an $N=1$ Super CFT (SCFT) of central charge 0 (at the tree level, respectively).

Moving on to $N=2$ SCFT, it is believed that one has to consider so-called semirigid Riemann surfaces of Distler and Nelson [DN91]. Not much of operadic properties of such objects is studied.

Problem 1. Study the notion of $N=2$ SCFT in physics literature, define the operad of semirigid super Riemann surfaces, and prove that an algebra over this operad is the same as an $N=2$ SCFT. This will generalize Huang's theorem [Hua97] which deals with the usual CFT.

2.2. **Universal Grassmannian.** This example belongs to A. S. Schwarz [Sch96], who suggested it as an operad governing a universal CFT in a certain sense.

Let $H = H_+ \oplus H_-$ be a separable Hilbert space split into the direct sum of two subspaces, along with a unitary involution K on H interchanging H_+ with H_- isomorphically. An example is the space $H = L^2(S^1)$ of square-integrable functions on the unit circle S^1. H_+ is the closure of the space of functions which may be extended holomorphically into the unit disk, and H_- is the closure of those functions which may be extended meromorphically inside the unit disk with a single pole at the origin. The involution K takes a function $f(z)$ to $f(1/z)/z$. The *universal Grassmannian* (as a set, although it is in fact an infinite dimensional complex manifold) is $\mathrm{Gr}(H) = \{V \subset H \mid \pi_+ : V \to H_+ \text{ is Fredholm and } \pi_- : V \to H_- \text{ is compact }\}$, where $\pi_\pm$ are the natural projections onto $H_\pm$. Recall that an operator is Fredholm, if it is bounded and has finite-dimensional kernel and cokernel, and compact, if the closure of the image of the unit ball is compact. The idea of the Grassmannian is to select a graspable set of subspaces V in H which differ not too much from H_+, but a lot from H_-.

For $n \geq 0$ there is a similar structure H_+^n, H_-^n, $K^{\oplus n}$ on the space H^n, and therefore, one can define the universal Grassmannian $\mathrm{Gr}(H^n)$ for each $n \geq 0$. For $n = 0$, it is just a point.

The observation of Schwarz is that the collection $\mathrm{Gr}(H^n)$, $n \geq 0$, is a modular operad. Indeed, for $1 \leq i < j \leq n$, we can define a mapping

$$\begin{aligned} \circ_{ij} : \mathrm{Gr}(H^n) &\to \mathrm{Gr}(H^{n-2}), \\ V &\mapsto p(f^{-1}(V)), \end{aligned}$$

where

$$\begin{aligned} f : H^{n-1} &\to H^n, \\ (x_1, \dots, x_i, \dots, \widehat{x}_j, \dots, x_n) &\mapsto (x_1, \dots, x_i, \dots, Kx_i, \dots, x_n), \end{aligned}$$

and

$$\begin{aligned} p : H^{n-1} &\to H^{n-2}, \\ (x_1, \dots, x_i, \dots, \widehat{x}_j, \dots, x_n) &\mapsto (x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_i, \dots, x_n). \end{aligned}$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, we can define a mapping

$$\circ_{ij} : \mathrm{Gr}(H^m) \otimes \mathrm{Gr}(H^n) \to \mathrm{Gr}(H^{m+n-2})$$

by composing the above $\circ_{ij}$ with the direct sum mapping

$$\begin{aligned} \mathrm{Gr}(H^m) \otimes \mathrm{Gr}(H^n) &\to \mathrm{Gr}(H^{m+n}), \\ (V_1, V_2) &\mapsto V_1 \oplus V_2. \end{aligned}$$

Proposition 2.1. *The above mappings $\circ_{ij}$ provide the collection $\mathrm{Gr}(H^n)$, $n \geq 0$, with the structure of a modular operad.*

Another remarkable fact, which makes this construction relevant to 2d quantum field theory, is that this operad structure is compatible with the Krichever map, i.e., there is a natural morphism from the modular operad of Riemann surfaces with holomorphic holes to the universal Grassmannian operad, see [Sch96] for more detail.

References

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