MATH 2243: LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
SOLUTION TO PROBLEM 5.4.15

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Solution: You have to memorize the general spring-mass equation $mx'' + cx' + kx = 0$, which in this case turns into

$$\frac{1}{2}x'' + 3x' + 4x = 0,$$

subject to the initial conditions $x(0) = 2, x'(0) = 0$. The equation rewrites as

$$x'' + 6x' + 8x = 0.$$

The characteristic equation is then

$$r^2 + 6r + 8 = 0,$$

with solutions $r_1 = -2$ and $r_2 = -4$. Thus, the theory of Section 5.3 on Homogeneous Equations with Constant Coefficients immediately gives you a general solution

$$x(t) = c_1 e^{-2t} + c_2 e^{-4t}.$$

From the initial condition $x(0) = 2$, we have $c_1 + c_2 = 2$, and from $x'(0) = 0$, we have $-2c_1 - 4c_2 = 0$, whence $c_2 = -2$ and $c_1 = 4$ and

$$x(t) = 4e^{-2t} - 2e^{-4t}.$$

In this you should immediately recognize overdamped motion, for example, realizing that out of the three types of motion, this one looks like the one decaying most rapidly, thereby, damping is most severe here.

The equation for the undamped position function is the one above, with $c = 0$, i.e.,

$$u'' + 8u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

It solves by using the characteristic equation

$$r^2 + 8 = 0,$$

whose solutions are $r = \pm i2\sqrt{2}$. Thus,

$$u(t) = c_1 \cos(2\sqrt{2}t) + c_2 \sin(2\sqrt{2}t),$$

and from the initial conditions we find $c_1 = 2$ and $c_2 = 0$, i.e.,

$$u(t) = 2\cos(2\sqrt{2}t).$$

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