MATH 2243 — FALL 2007 FINAL EXAM
DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA, MINNEAPOLIS

NAME: ____

ID NUMBER: ____________________

(1) Do not open this exam until you are told to begin.
(2) This exam has 13 pages including this cover and two intentionally blank pages for your use. There are 10 problems total. You have 3 hours.
(3) No notes or books are permitted.
(4) Only non-graphing calculators are permitted.
(5) Please turn off all cell phones.
(6) Place your ID card on your desk for inspection.
(7) Good luck!

PROBLEM	POINTS	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. Determine whether the following statements are **true** or **false**. If you state the statement is **false** give a counterexample that demonstrates your claim. *You will be scored +2 points for a complete correct answer. 0 for no answer and −2 for an incorrect answer. Thus guessing will be penalized.*

F (a) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices then $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$. $= B^{-1}A^{-1}$

F (b) Given a collection of linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in a vector space V, any vector $\mathbf{x}$ can be expressed as $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. no because $\{v_1 .. v_k\}$ may not span V

T (c) If two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, then $\det \mathbf{A} = \det \mathbf{B}$.

F (d) Suppose V is a vector space, $S \subseteq V$ is a subspace of V, $\mathbf{v} \in S$ and $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in V$. Then $\mathbf{a}, \mathbf{b} \in S$. $S = \{t\begin{bmatrix}1\\1\end{bmatrix}\} < \mathbb{R}^2$ take $v = \begin{bmatrix}1\\1\end{bmatrix}$ $a = \begin{bmatrix}1\\0\end{bmatrix}$ $b = \begin{bmatrix}0\\1\end{bmatrix}$

F (e) For all values of the coefficients k and c the following system has only one solution.

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & c \\ 3 & 2 & 1 & 5 \\ 0 & 1 & k & 3 \end{array}\right]$$

no: when $k=1$. $c=6$ $\Rightarrow \infty$ solutions

$k=1$ $c \neq 6 \Rightarrow$ no solution

2. Solve the following differential equation:

$$\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3 - y)}.$$

You may leave your answer in implicit form.

$$\frac{2y^3-y}{y^5}\, y' = \frac{x-1}{x^2}$$

$$\left(2\,\frac{1}{y^2} - \frac{1}{y^4}\right) y' = \frac{1}{x} - \frac{1}{x^2}$$

$$-\frac{1}{y} + \frac{1}{3} y^{-3} = \ln|x| + \frac{1}{x} + C$$

3. Consider a population $P(t)$ satisfying the logistic equation $P'(t) = aP - bP^2$, where $\alpha = a$ is the constant birth rate per month per individual, and $\beta = bP$ is the death rate per month per individual. Assume that the initial population is 1000 individuals and there are 50 births and 20 deaths per month occurring at time $t = 0$.

(a) Calculate the constants a and b from the data.

(b) What is the maximum population which can be attained?

(c) How many months does it take for the population to reach 80% of the limit population?

(a) $a \cdot 1000 = 50 \quad a = 0.05$

$b \cdot 1000^2 = 20$

$b = 2\times 10^{-5}$

(b) $P' = 0 = aP - bP^2 = P(a - bP)$

$P = 0$ or $0.05 = 2\times10^{-5} \cdot P \qquad P = \frac{0.05}{2} \times 10^5 = \frac{5000}{2} = 2500$

(c) $80\% \times 2500 = 2000$

$\frac{dP}{dt} = aP - bP^2 = P(a - bP) = P(0.05 - 2\times10^{-5}P) = 2\times10^{-5}P(2500 - P)$

$\frac{1}{P(2500-P)} P' = 2\times10^{-5}$

$\frac{1}{2500}\left(\frac{1}{P} + \frac{1}{2500-P}\right)P' = 2\times10^{-5}$

$\ln|P| - \ln|2500-P| = 0.05t + C$

$\ln 1000 - \ln 2500 = C = \ln\frac{10}{25} = \ln\frac{2}{5}$

$\ln|2000| - \ln 500 = 0.05t + \ln\frac{2}{5}$

$\ln 4 = 0.05t + \ln 2/5 \qquad 0.05t = \ln 4 \times \frac{5}{2} = \ln 10$

$t = 20 \ln 10$

X

4. Consider the differential equation $dx/dt = -x^2 + kx - 1$ containing the parameter k. Determine the number and stability or unstability of the critical points depending on the value of k. Construct the bifurcation diagram.

5. Let

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

be a 3×3 matrix. Find a collection of linearly independent vectors such that solution space corresponding to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is the set of linear combinations of those vectors.

That means to find a basis of the solution space of $Ax=0$

$$\begin{bmatrix} 0 & 2 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} & & \uparrow \\ & & \text{free} \end{array}$$

let $x_3 = t$ $\Rightarrow 3x_1 - t = 0$ $x_1 = t/3$

$x_2 + x_3 = 0$ $x_2 = -t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t/3 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -1 \\ 1 \end{bmatrix}$$

Answer: $\left\{ \begin{bmatrix} \frac{1}{3} \\ -1 \\ 1 \end{bmatrix} \right\}$

6. Find the eigenvalues and a full set of linearly independent eigenvectors of the two following matrices. Determine which of them is diagonalizable and find a diagonalizing matrix S and a diagonal matrix D such that it is equal to $S^{-1}DS$.

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 3 & 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 3 & -3 & 5 \end{bmatrix}.$$

$$|B-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 3 & 2 & 5-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 3 & 5-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2-6\lambda+5+3)$$

$$= (2-\lambda)(\lambda^2-6\lambda+8)$$

$$= (2-\lambda)(\lambda-4)(\lambda-2)$$

$\lambda = 2, 2, 4$ $\quad \lambda = 2 \quad B-\lambda I = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & 2 & 3 \end{pmatrix}$ eigenspace 1-dim

B is NOT diagonalizable

$|C-\lambda I| = (2-\lambda)(\lambda^2-6\lambda+5+8)$ $\quad \lambda = 2, 2, 4$

$C-2I = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & -3 & 3 \end{pmatrix}$ $\quad (C-2I)v=0 \Rightarrow \begin{matrix} -v_1+v_2-v_3=0 \\ 3v_1-3v_2+3v_3=0 \end{matrix}$ same eqn

$v_2 = s \quad v_3 = t \qquad v_1 = s-t \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

eigenvectors

$\lambda = 4$

$C-4I = \begin{pmatrix} -3 & 1 & -1 \\ 0 & -2 & 0 \\ 3 & -3 & 1 \end{pmatrix} \rightarrow$ eigenvector $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

$C = S^{-1}DS \qquad S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

7. Find the general solution to the differential equation:

$$y^{(3)} - 2y'' + y' = x^2 + xe^x.$$

char poly $t^3 - 2t^2 + t = 0$

$t\,(t-1)^2 = 0 \quad t = 0, 1, 1$

$\rightarrow y_h = c_1 e^0 + c_2 e^x + c_2 e^x x$

$y_p = (Ax+B)e^x \cdot x^2 + Cx^3 + Dx^2 + E$

8. Apply the eigenvalue method to solve the following system of differential equations:

$$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= -5x_1 - 3x_2 - x_3 \\ x_3' &= 5x_1 + 5x_2 + 3x_3 \end{aligned}$$

$$X' = \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{pmatrix}}_{A} X$$

$$|A - \lambda I| = (3-\lambda)^2(-3-\lambda) - 5 - 25 + (3+\lambda)5 + (3-\lambda)(5+5)$$

$$= (3-\lambda)^2(-3-\lambda) - 30 + 15 + 5\lambda + 30 - 10\lambda$$

$$= (3-\lambda)^2(-3-\lambda) + 15 - 5\lambda$$

$$= (3-\lambda)(\lambda^2 - 9 + 5) = (3-\lambda)(\lambda^2 - 4)$$

$\lambda = 3, 2, -2$

$\lambda = 3$ $\quad A - 3I = \begin{pmatrix} 0 & 1 & 1 \\ -5 & -6 & -1 \\ 5 & 5 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$(A - 3I)v = 0 \Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \quad \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda = 2$ $\quad A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ -5 & -5 & -1 \\ 5 & 5 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

eigenvector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\lambda = -2$ $\quad A + 2I = \begin{pmatrix} 5 & 1 & 1 \\ -5 & -1 & -1 \\ 5 & 5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

general soln $= c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-2t}$

9. Solve the system of linear equations

$$\begin{aligned} 3x_1 + x_2 + x_3 + 6x_4 &= 14 \\ x_1 - 2x_2 + 5x_3 - 5x_4 &= -7 \\ 4x_1 + x_2 + 2x_3 + 7x_4 &= 17 \end{aligned}$$

Please do it by yourself...

10.

(a) Find the Laplace transform of $f(t) = \sin^2(t/2)$.

(b) Find the inverse Laplace transform of $F(s) = \dfrac{1}{2s^2(s^2+1)}$.

1(a) $1 - 2\sin^2\left(\frac{t}{2}\right) = \cos t$

$\frac{1-\cos t}{2} = \sin^2\frac{t}{2}$

$\mathcal{L}\{f\} = \mathcal{L}\left\{\frac{1-\cos t}{2}\right\}$

$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+1}$

1(b) $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$

$\mathcal{L}^{-1}(F) = \frac{1}{2}(t - \sin t)$

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