Math 2243
Spring 2006
Final

Name: Solution ??
I.D.:
T.A.:
Discussion section:

For you reference only, may contain error !!

This is a closed book, closed notes exam. Only scientific calculators are allowed. There are 12 problems on the exam. The number of points for each problem is indicated. Please show your work. Good Luck!

1. (10 pts) Determine if the functions $\{ t^2+1,\ 2t^2-2,\ 3t+15 \}$ are linearly independent or not.

l. indep

If $c_1(t^2+1)+c_2(2t^2-2)+c_3(3t+15)=0$

$(c_1+2c_2)t^2+3c_3t+(c_1-2c_2+15c_3)=0$

$c_1+c_2=0$
$3c_3=0$
$c_1-2c_2+15c_3=0$
$\Rightarrow c_1=c_2=c_3=0$

thus l. indep

2. (10 pts) Find the general solution of the differential equation

$$t\frac{dy}{dt} + 2y = 3t, \qquad t > 0.$$

$y' + \frac{2}{t} y = 3 \quad t > 0$

$p(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2$

$t^2 y' + 2ty = 3t^2$

$(t^2 y)' = 3t^2$

$t^2 y = \int 3t^2 dt = t^3 + C$

$\boxed{y = t + \frac{C}{t^2}}$ C : arbitrary real number

3. (15 pts) (a) Convert the equation $y'' - 6y' + 8y = 0$ to a first-order system.

let $x = y'$

$$\begin{cases} x' - 6x + 8y = 0 \\ y' = x \end{cases}$$

(b) Transform the following system into a second-order differential equation.

$$x_1' = 3x_1 + x_2$$
$$x_2' = x_1 + 5x_2$$

Try to clear x_2

$$x_1'' = 3x_1' + x_2' = 3x_1' + x_1 + 5x_2 = 3x_1' + x_1 + 5(x_1' - 3x_1)$$

$$= 8x_1' - 14x_1$$

$$\boxed{x_1'' = 8x_1' - 14x_1}$$

4. (15 pts) Find the solution of the initial value problem in explicit form.

$$\frac{dy}{dt} = 2ty^2 - 3y^2, \qquad y(0) = -\frac{1}{6}.$$

$\frac{1}{y^2} y' = 2t - 3 \quad y(0) = \frac{-1}{6}$

$-\frac{1}{y} = t^2 - 3t + c$

$y = \frac{-1}{t^2 - 3t + c} \qquad y(0) = -1/6$

$\Rightarrow \frac{-1}{6} = \frac{-1}{c} \qquad c = 6$

$\boxed{y = \frac{-1}{t^2 - 3t + 6}}$

5. (20 pts) Suppose it is $75°F$ in your house when your air conditioner breaks down at noon. The outside temperature is $90°F$. You notice that after one hour the inside temperature reaches $80°F$. Find a formula describing the inside temperature at any time t.

$$\frac{dT}{dt} = k(T-A)$$

$$\frac{1}{T-A}\frac{dT}{dt} = k$$

$$\ln|T-A| = kt + C$$

$$T-A = \pm e^{kt+c}$$

$$T = A \pm e^{kt+c}$$

$$T(0) = 75 \quad A = 90$$

$$T(1) = 80$$

$$75 = T(0) = 90 \pm e^{c} \quad \Rightarrow \pm e^{c} = -15$$

$$T = 90 - 15e^{kt}, \; 80 = T(1) = 90 - 15e^{k}$$

$$10 = 15e^{k} \quad \frac{2}{3} = e^{k}$$

$$T = 90 - 15 \cdot \left(\frac{2}{3}\right)^{t}$$

6. (20 pts) The matrix given below has eigenvalues $\lambda = -1, 2$. Please find the corresponding eigenvectors, and determine the dimension of each eigenspace.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

$\lambda = -1$ $\quad A + I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad (A+I)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$ $\quad v_1 + \overset{s}{v_2} + \overset{t}{v_3} = 0$

$v_1 = -t - s$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ eigenvectors

the dimension of the eigenspace asso w/ $\lambda = -1$ is 2

$\lambda = 2$ $\quad A - 2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad v_1 - 2v_2 + v_3 = 0$

$v_2 - \underset{t}{v_3} = 0$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftarrow$ eigenvectors if $t \neq 0$

dim of eigenspace asso w/ $\lambda = 2$ is 1

7. (20 pts) Determine the kernel, image, nullity, and rank of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\mathbf{v}) = A\mathbf{v}$, where

not covered

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 6 & 6 \end{bmatrix}$$

8. (20 pts) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$A = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$ $|A - \lambda I| = \begin{bmatrix} 3-\lambda & 4 \\ -4 & -5-\lambda \end{bmatrix} = \lambda^2 + 2\lambda - 15 + 16$

$= (\lambda+1)^2$

$\lambda = -1, -1$

$A + I = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}$ eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ← only one, so have to find generalized eigenvector

$\begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$4v_1 + 4v_2 = 1$ $-4v_1 - 4v_2 = -1$ many solutions, take $\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$

so $\begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to 0$ is a chain

$x = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right) e^{-t}$

9. (20 pts) (a) Let $A = \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}$. Find an invertible matrix P so that $P^{-1}AP$ is a diagonal matrix.

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 5 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8 + 5 = (\lambda-3)(\lambda+1)$$

$\lambda = 3$ $\quad A - 3I = \begin{bmatrix} 1 & 5 \\ -1 & -5 \end{bmatrix}$ $\quad$ eigenvector $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$\lambda = -1$ $\quad A + I = \begin{bmatrix} 5 & 5 \\ -1 & -1 \end{bmatrix}$ $\quad$ " $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $P^{-1}AP = D$ $\quad P = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}$

$A = PDP^{-1}$

(b) Decouple the linear system

$$\mathbf{x}' = \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix} \mathbf{x}.$$

Then solve the system.

$$x = c_1 \begin{bmatrix} 5 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

10. (20 pts) Solve the initial value problem

$$X'' + 4X = 6\sin t, \qquad X(0) = 0, \quad X'(0) = 1.$$

$X_c = C_1 \sin 2t + C_2 \cos 2t$

Let $x_p = A \sin t$

$x_p'' = -A\sin t$

$x_p'' + 4x_p = -A\sin t + 4A\sin t = 3A\sin t \overset{\text{need}}{=} 6\sin t$

$A = 2$

$X = X_c + X_p = C_1 \sin 2t + C_2 \cos 2t + 2\sin t$

$0 = X(0) = C_2$

$1 = X'(0) = 2C_1 \cos 2\cdot 0 + 2\cos 0 = 2C_1 + 2$

$C_1 = \frac{-1}{2}$

$X = -\frac{1}{2}\sin 2t + 2\sin t$

11. (20 pts) Find the general solution of the equation

$$y'' - 5y' + 4y = 2e^t + 6.$$

$y_c = C_1 e^{4t} + C_2 e^{t}$

$y_p = Ate^t + B$

$y_p' = Ae^t + Ate^t$

$y_p'' = 2Ae^t + Ate^t$

$y_p'' - 5y_p' + 4y_p = Ae^t(2-5) + (1-5+4)Ate^t + 4B$

$= 2e^t + 6$

$\Rightarrow B = \frac{3}{2} \quad -3A = 2 \quad A = -\frac{2}{3}$

$y = C_1 e^{4t} + C_2 e^t - \frac{2}{3} te^t + \frac{3}{2}$

12. (10 pts) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

.... Youcan do it .. det = 0

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \overset{(-1)R_2+R_3}{=} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} \underset{\substack{(-1)R_1 \\ +R_2}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$