Math 2243
Spring 2008
FINAL EXAM

Name (Print) Solution

Signature NOT guaranTee to be 100% correct

Recitation Instructor________ Section________ I.D.#________

READ AND FOLLOW THESE INSTRUCTIONS:
This booklet contains 13 pages,including this cover page and a scratch page. Check to see if any are missing. PRINT on the upper right-hand corner all the requested information, and sign your name. Put your initials on the top of every page, in case the pages become separated. Textbooks, notes and calculators **are not permissible**. Do your work in the blank spaces and back of pages of this booklet. Show all your work!

There are 10 machine-graded problems, each worth 15 points. There are 6 hand-graded problems of varying worth for 150 points. This gives a total of 300 points.

INSTRUCTIONS FOR MACHINE-GRADED PART (Questions 1-10):
You MUST use a soft pencil (No. 1 or No 2) to answer this part. Do not fold or tear the answer sheet, and carefully enter all the requested information according to the instructions you receive. **DO NOT MAKE ANY STRAY MARKS ON THE ANSWER SHEET.** When you have decided on a correct answer to a given question, circle the answer in this booklet and blacken completely the corresponding circle in the answer sheet. If you erase something, do so completely. Each question has a correct answer. If you give two different answers, the question will be marked wrong. There is no penalty for guessing, but if you don't answer a question, skip the corresponding line in the answer sheet. Go on to the next question.

INSTRUCTIONS FOR THE HAND-GRADED PART (Questions 11-16):
SHOW ALL WORK. Unsupported answers will receive little credit.

Notice regarding the machine graded sections of this exam: Either the student or the School of Mathematics may for any reason request a regrade of the machine graded part. All regrades will be based on responses in the test booklet, and not on the machine graded response sheet. Any problem for which the answer is not indicated in the test booklet, or which has no relevant accompanying calculations will be marked wrong on the regrade. *Therefore work and answers must be clearly shown on the test booklet.*

AFTER YOU FINISH BOTH PARTS OF THE EXAM: Place the answer sheet between two pages of this booklet (make a sandwich), with the side marked "GENERAL PURPOSE ANSWER SHEET" facing **DOWN**. Have your ID card in your hand when turning in your exam.

Multiple choice part ________ Hand-graded part ________

Problem	
11	
12	
13	
14	
15	
16	
Subtotal	

Total ________

1. Let y be the solution of the IVP

$$y' + 2y = x, \quad y(0) = \frac{3}{4}$$

then $y(1) =$

$\rho(x) = e^{\int 2\,dx} = e^{2x}$

(a) $\frac{1}{2} + \frac{3}{4e^2}$

(b) $\frac{1}{4} + e^2$

$\Rightarrow e^{2x}y' + e^{2x} \cdot 2y = xe^{2x}$

$(e^{2x}y)' = xe^{2x}$

✓ (c) $\frac{1}{4} + \frac{1}{e^2}$

(d) $\frac{1}{2} + \frac{3}{4}e^2$

integrate both sides

(e) none of the above.

$e^{2x} \cdot y = \int \underset{u\ v'}{xe^{2x}}dx = \underset{u\ v}{x\frac{1}{2}e^{2x}} - \int \underset{u'v}{\frac{1}{2}e^{2x}}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}$

$y(0) = 3/4 \Rightarrow 3/4 = -1/4 + c \Rightarrow c = 1$

$y(1) = \frac{1}{2} - \frac{1}{4} + 1 \cdot e^{-2}$

2. In a cold but calm day, a thermometer is taken from an inside room with temperature 70°F to the outside. After one minute the thermometer reads 40°F and after two minutes the reading is 20°F. What is the air temperature outside?

✓

(a) −20°F

$\frac{dT}{dt} = k(T-A) \quad T(0) = 70 \quad T(1) = 40 \quad T(2) = 20$

(b) 0°F

$\frac{1}{T-A}\frac{dT}{dt} = k \Rightarrow \int \frac{1}{T-A}d(T-A) = \int k\,dt = kt + C$

(c) $\left(\frac{\ln(70-40)}{\ln(40-20)}\right)^o$F

(d) $(\ln \frac{2}{3})^o$F

$\ln|T-A| = kt + c$

$T - A = e^{kt+c}$

(e) none of the above.

$T = e^{kt}e^c + A$

$T(0) = e^c + A = 70 \qquad e^c = 90$

$T(1) = e^ce^k + A = 40 \qquad e^ce^k = 60$

$T(2) = e^ce^{2k} + A = 20 \qquad e^ce^k = 40$

$(40-A)^2 = (e^ce^k)^2 = e^c \cdot e^c \cdot e^{2k} = (70-A)(20-A)$

$1600 - 80A + A^2 = 1400 - 90A + A^2 \quad 200 = -10A \quad A = -20$

X 3. Consider the salmon population in a fish farm. Suppose it is modeled by the equation

$$\frac{dP}{dt} = P(40 - P) - h$$

where P represents the fish population and h is the constant harvesting rate. Suppose $P(0)$ is very large. What is the maximum sustainable harvesting rate h?

(a) 20

(b) 40

(c) 200

(d) 400

(e) none of the above.

X 4. What is the interval of existence of the solution of the IVP

$$y' = t^2 y^2, \quad y(0) = 3?$$

(a) The whole real line.

(b) The whole real line except the point $t = 0$.

(c) The whole real line except the point $t = 1$.

(d) No solution exist in any interval.

(e) none of the above.

X

5. If the Wronskian of a set of differentiable functions $\{f_1, \ldots, f_n\}$ vanishes at a point, then

(a) the Wronskian must vanish identically.

(b) the set of functions $\{f_1, \ldots, f_n\}$ is linearly dependent.

(c) the set of functions $\{f_n, \ldots, f_n\}$ is linearly independent.

✓(d) there is not enough information to conclude that the set $\{f_1, \ldots, f_n\}$ is linearly dependent.

(e) none of the above.

in fact $\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

not covered

X 6. Which of the following statements concerning the functions $f(x) = x^2$ and $g(x) = x|x|$ on $\mathbf{R}$ is false?

(a) $\{f, g\}$ is linearly independent.

(b) The Wronskian $W[f, g]$ vanishes identically.

(c) $\{f, g\}$ is not a basis for the set of all polynomials of degree ≤ 2.

(d) Span$\{f, g\}$ does not contain the constant function 1.

✓(e) none of the above.

NOT convered

7. The equation $y'' - 2y' + 3y = 0$ is equivalent to the system $\mathbf{x}' = A\mathbf{x}$ where $A =$

$y'' - 2y' + 3y = 0$

$x = y'$

$x' - 2x + 3y = 0$

$y' = x$

$x' = -3y + 2x$

(a) $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$

✓ (b) $\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$

(e) none of the above.

X 8. The trajectories of the system

$$\begin{cases} x' &= y(1 + x^2 + y^2) \\ y' &= x(1 + x^2 + y^2) \end{cases}$$

are

(a) circles

(b) ellipses

(c) hyperbolas

(d) parabolas

(e) none of the above.

X 9. Let $A = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}$. Then $e^A =$

(a) $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

(b) $\frac{1}{2}\begin{pmatrix} 3e^3 - e^5 & -e^3 + e^5 \\ 3e^3 - 3e^5 & -e^3 + 3e^5 \end{pmatrix}$

(c) $\begin{pmatrix} e^3 & 0 \\ 0 & e^5 \end{pmatrix}$

(d) $\frac{1}{2}\begin{pmatrix} 3e^3 - e^5 & 3e^3 - 3e^5 \\ -e^3 + e^5 & -e^3 + 3e^5 \end{pmatrix}$

(e) none of the above.

X 10. The critical point $(0, 0)$ for the system

$$\begin{cases} x' &= x - 3y + 2xy \\ y' &= 4x - 6y - xy \end{cases}$$

is a

(a) stable improper node

(b) unstable improper node

(c) center

(d) unstable saddle point

(e) none of the above.

Handgraded part

11.(25 pts) The rate at which the volume of a mothball evaporates from solid to gas is proportional to the surface area of the ball. Suppose it has been observed that a mothball of radius $\frac{1}{2}$ inch evaporates to leave a ball of radius $\frac{1}{4}$ inch at the end of six months. After how many more months will it disappear completely?

$\frac{dV}{dt} = -kA$ $\quad V = \frac{4}{3}\pi r^3 \quad A = 4\pi r^2$

$$\frac{d\left(\frac{4}{3}\pi r^3\right)}{dt} = -k \cdot 4\pi r^2$$

$$4\pi r^2 \frac{dr}{dt} = -k\, 4\pi r^2$$

$$\frac{dr}{dt} = -k$$

$$r = -kt + C$$

$$r(0) = C = \frac{1}{2}$$

$$r(6) = -6k + C = \frac{1}{4} \quad k = \frac{1}{24}$$

$r = -\frac{1}{24}t + \frac{1}{2}$ $\quad r=0$ when $t = 12$ months

6 more months

12.(20 pts) Find a basis for $\mathbf{R}^3$ that contains the vectors $(1,2,2)$ and $(2,3,3)$.

Since the last two components of $\begin{bmatrix}1\\2\\2\end{bmatrix}$ & $\begin{bmatrix}2\\3\\3\end{bmatrix}$ are the same

If we pick $\begin{bmatrix}0\\0\\1\end{bmatrix}$, it's indep of $\begin{bmatrix}1\\2\\2\end{bmatrix}$ & $\begin{bmatrix}2\\3\\3\end{bmatrix}$

check: $\begin{pmatrix}1&2&2\\2&3&3\\0&0&1\end{pmatrix} \to \begin{pmatrix}1&2&0\\2&3&0\\0&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

so $\begin{bmatrix}1\\2\\2\end{bmatrix}$ $\begin{bmatrix}2\\3\\3\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ are l. indep

X 13.(25 pts) A 32 lb weight stretches a spring (hanging from the ceiling) 6 inches. The weight moves through a medium that provides α lb of resistance for every foot per second of velocity. Determine the value of α for which the system will exhibit oscillatory motion.

14.(20 pts) Find a general solution of the system

$$x' = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} x$$

Let $A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = (\lambda^2 - 8\lambda + 15) + 1 = (\lambda-4)^2$

$\lambda = 4, 4$

$A - 4I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ $(A-4I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$(A-4I)v = 0$ $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ $-v_1 - v_2 = 0$ eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

since there's only one l. indep eigenvector, need a chain $v_2 \to v_1 \to 0$ for $\lambda = 4$

$(A-4I)^2 v_2 = 0$ pick $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_1 = (A-4I)v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq 0$

so $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ form a length 2 chain

general solution is

$$X(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t}$$

15.(30 pts) Solve the initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t\cos 2t \\ t\sin 2t \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

X 16.(30 pts) Find all critical points of the following system and determine the type and stability of each:

$$\begin{cases} x' &= xy - 2 \\ y' &= x - 2y \end{cases}$$

Scratch