MATH 2243: LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
SAMPLE MIDTERM EXAM III

You may not use notes, books, etc. Only the exam paper, a pencil or pen, and a basic or scientific calculator may be kept on your desk during the test. Unless stated otherwise, please show all of your work and justify your answers in order to receive full credit.

Good luck!

Date: April 24, 2014; Solution to #4 and Answer to #5 added on April 30, 2014.

Problem 1. (7 points) A body with mass $m = 4$ kg is placed on a table and attached to the end of a spring on one side and a dashpot on the other side. The spring is stretched 2 meters by a force of 50 N and the dashpot provides $c = 12$ N of resistance per each meter per second of velocity. The body is set in motion with initial position $x(0) = 4$ m and initial velocity $v(0) = 0$ m/s. Find the position function of the body as well as the amplitude and frequency.

Problem 2. (7 points) Find a particular solution of the equation

$$y'' - 4y' - 5y = 4e^{5x}.$$

Problem 3. (6 points) Consider the following matrix

$$A = \begin{bmatrix} 2+a & -3 \\ 1 & 1 \end{bmatrix}.$$

For what values of a does there exist a matrix P such that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}?$$

[Hint: What does the condition tell you about the eigenvalues of A?]

Problem 4. (10 points) Find the particular real-valued solution of the system

$$\begin{aligned} x_1' &= -x_1 + 3x_2, \\ x_2' &= -3x_1 - x_2 \end{aligned}$$

with initial conditions

$$x_1(0) = 3, \qquad x_2(0) = -1.$$

Solution: Find the eigenvalues of the matrix:

$$\begin{vmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 10 = 0,$$

whence

$$\lambda = -1 \pm 3i.$$

Choose, say, $\lambda = -1 + 3i$ and find an associated complex eigenvector:

$$\mathbf{v} = (-i, 1).$$

A complex-valued solution will then be

$$e^{-t}(\cos 3t + i \sin 3t) \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Its real and imaginary parts produce two lineraly independent real-valued solutions

$$\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{bmatrix} -\cos 3t \\ \sin 3t \end{bmatrix}.$$

A linear combination of them will be a vector-function whose components are

$$\begin{aligned} x_1 &= e^{-t}(c_1 \sin 3t - c_2 \cos 3t), \\ x_2 &= e^{-t}(c_1 \cos 3t + c_2 \sin 3t). \end{aligned}$$

Substituting the initial conditions, we get $c_1 = -1$ and $c_2 = -3$. Thus,

$$\begin{aligned} x_1 &= e^{-t}(-\sin 3t + 3 \cos 3t), \\ x_2 &= -e^{-t}(\cos 3t + 3 \sin 3t). \end{aligned}$$

Problem 5. (10 points) Find a general real-valued solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}.$$

Answer:

$$c_1 e^{5t}(1,0,0) + c_2 e^{2t}(0,3,0) + c_3 e^{2t}(0,3t,1).$$