MATH 4281: INTRODUCTION TO MODERN ALGEBRA SAMPLE MIDTERM TEST I (WITH SELECTED SOLUTIONS)

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You may not use a calculator, notes, books, etc. Only the exam paper and a pencil or pen may be kept on your desk during the test.

Good luck!

Problem 1. Prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

Problem 2. Prove that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n.$$

Problem 3. Find $\gcd(210,48)$ in two different ways and find two integers s and t such that $210s+48t=\gcd(210,48)$.

Problem 4. Find an integer x such that

$$\begin{aligned} x &\equiv 2 \pmod 5,\\ 3x &\equiv 1 \pmod 8. \end{aligned}$$

Problem 5. Find $[20877^{24}]$ and $[20878^{24}]$ in $\mathbb{Z}_{16}$. [Hint: Use Euler's theorem.]

Problem 6. Is $x^3-17x+2$ irreducible in $\mathbb{Z}[x]$, $\mathbb{Q}[x]$, $\mathbb{R}[x]$, and $\mathbb{C}[x]$? Explain why.

Solution: First of all, note that a polynomial of degree 3 factors in $K[x]$, if and only if it factors as $(ax+b)(cx^2+dx+e)$ in $K[x]$. If $K=\mathbb{Z}$, then since $ac=1$, $a=\pm 1$, and we can assume without loss of generality that $a=1$ by moving the -1 over to the second factor. If K is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, we can move a over to the second factor and conclude that a polynomial of degree 3 is reducible in $K[x]$, if and only if it has a factor $x-b$. Thus, for any K, a polynomial of degree 3 is reducible in $K[x]$, if and only if it has a root in K.

In $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$: By one of the exercises, a rational root r/s in the reduced form will have $r|2$ and $s|1$. Thus we can assume $s=1$ and $r=\pm 1$ or ± 2. Plug these into the polynomial, and we see that none of these is a root. Thus it has no rational and in particular no integral roots.

In $\mathbb{R}[x]$: Call our polynomial $p(x)$. Obviously, $p(-10000)<0$, whereas $p(10000)>0$. Since $p(x)$ is continuous, by the Intermediate Value Theorem, it assumes all intermediate values, such as 0. Thus, it has a root, and therefore will be reducible over $\mathbb{R}$.

In $\mathbb{C}[x]$ every polynomial of degree at least 1 has a root and will thereby be reducible.

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Problem 7. List all motion symmetries of a regular pentagon.

Solution: Put the pentagon in the xy plane with its centroid at the origin. Take the rotation r of the pentagon about the z axis by $360°/5 = 72°$, counterclockwise if you look from the above, and the $180°$ rotation a about the axis passing through one of the vertices and the midpoint of the oppoisite side. These are symmetries of the pentagon. One can generate more symmetries by these:

$$\{e, r, r^2, r^3, r^4, a, ra, r^2a, r^3a, r^4a\}.$$

The rotations e, r, r^2, r^3, and r^4 are counterclockwise rotations by $0°$, $72°$, $2 \cdot 72°$, $3 \cdot 72°$, and $4 \cdot 72°$, respectively, and rotate the space by different angles less than $360°$. Thus, they must be pairwise distinct. If $r^k a = r^m a$, then by multiplying by a^{-1} on the right, we get $r^k = r^m$ with $0 \le k, m \le 4$, and we saw that all these are different for different k and m. Finally, $r^k \ne r^m a$, because $r^m a$ flips the pengaton over and r^k does not. Thus, we have got at least 10 distinct symmetries.

On the other hand, any symmetry of the pentagon must map a particular vertex, call it A_1, to another vertex, for which there are 5 different choices, and the counterclockwise adjacent vertex A_2 will be mapped to either of the nearby vertices to the vertex where A_1 went to. If we know where A_1 and A_2 go, we will know where the remaining vertices of the pentagon will go. There are $5 \cdot 2 = 10$ choices for that. Thus, there should be not more than 10 symmetries, and the above list shows all of them.