MATH 4281: INTRODUCTION TO MODERN ALGEBRA SAMPLE MIDTERM TEST II

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You may not use a calculator, notes, books, etc. Only the exam paper and a pencil or pen may be kept on your desk during the test.

Good luck!

Problem 1. Let

$$\pi = (12)(58)(346)(52)(41)(37)(67).$$

(1) Write π as a product of disjoint cycles.
(2) Is $\pi \in A_8$?
(3) What is the maximum possible order of an element in S_8?

Problem 2. Prove that

$$H = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

is a subgroup of $G = \mathrm{SL}(2, \mathbb{R})$. (Be sure to show that H is a *subset* of G.)

You may find the following trig identities useful:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A, \quad \sin(A - B) = \sin A \cos B - \sin B \cos A,$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \quad \cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Problem 3. Let G be a group and suppose that $|G| = pq$ where p and q are primes. Prove that every proper subgroup of G is cyclic.

Problem 4. Let $G = \mathbb{Z}_{20}$, and $H =< [4] >$. List the distinct left cosets of H in G.

Problem 5. Give the definition of a zero divisor. Give an example of such.

Problem 6. Give the definition of a field as a ring with certain conditions. Which of the following are (1) fields, (2) rings, or (3) neither:

$$\mathbb{C}, \quad \mathbb{Z}, \quad \mathrm{Mat}_2(\mathbb{R}), \quad \mathrm{GL}(2, \mathbb{R}), \quad \mathbb{Z}_{35}, \quad \mathbb{Z}_{17}, \quad \mathbb{Z}_{200,000}, \quad K[x], \quad K[x, x^{-1}]?$$

Explain only why one is not a ring or not a field.

Answer: $\mathbb{C}$ is a field (each nonzero complex number has an inverse) $\mathbb{Z}$ is a ring (two ring operations, but the units are only ± 1), $\mathrm{Mat}_2(\mathbb{R})$ is a ring (two ring operations, but not every matrix is invertible), $\mathrm{GL}(2, \mathbb{R})$ is neither (not a ring and thereby a field, because closed only under multiplication), $\mathbb{Z}_{35}$ is a ring (two ring operations, but has zero divisors: [5] and [7]), $\mathbb{Z}_{17}$ is a field (two ring operations, and every nonzero element is invertible), $\mathbb{Z}_{200,000}$ is a ring (two ring operations, but [2] is a zero divisor), $K[x]$ is a ring (two ring operations, but the units are only constant polynomials), $K[x, x^{-1}]$? is a ring (two ring operations, but the units are only ax^n, $a \in K$, $n \in \mathbb{Z}$).

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Problem 7. Let G and H be groups and $\phi : G \to H$ a homomorphism. Prove that $\ker(\phi)$ is normal in G.