MATH 4281: INTRODUCTION TO MODERN ALGEBRA SAMPLE MIDTERM TEST III, WITH SELECTED SOLUTIONS

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You may not use a calculator, notes, books, etc. Only the exam paper and a pencil or pen may be kept on your desk during the test.

Good luck!

Problem 1. Let $G = \mathbb{Z}_{20}$, and $H =< [4] >$.

(1) What is the order of $[6] + H$ in G/H?
(2) Is G/H isomorphic to $\mathbb{Z}_4$ or the group of symmetries of the rectangle? Explain your answer.

Problem 2. Let $G = D_4 \times D_4$ and $H = \langle (r, e) \rangle$, the subgroup generated by the element $(r, e) \in D_4 \times D_4$, where r is a 90-degree counterclockwise rotation of the square about the centroid axis and e is the nonmotion. Let j be the flip about the axis passing through the opposite vertices of the square.

(1) Show that H is normal and calculate the order of the quotient group G/H.
(2) What is the order of $(r^3, j)H$ in G/H?

Problem 3. (1) Find all abelian groups of order 20 up to isomorphism.
(2) Does every abelian group of order 20 have an element of order 4?
(3) Does every abelian group of order 20 have an element of order 5?

Problem 4. Let G_1 and G_2 be groups and H_1, H_2 normal subgroups of G_1 and G_2, respectively. Prove that

$$(G_1 \times G_2)/(H_1 \times H_2) \cong G_1/H_1 \times G_2/H_2.$$

[Hint: Use one of the homomorphism theorems.]

Solution: Define a homomorphism

$$\phi : G_1 \times G_2 \to G_1/H_1 \times G_2/H_2$$

by the formula

$$\phi(g_1, g_2) := (g_1 H_1, g_2 H_2).$$

Check that ϕ is a surjective ring homomorphism. Its kernel $\ker \phi$ is the set of pairs (g_1, g_2) such that $(g_1 H_1, g_2 H_2) = (H_1, H_2)$, which means $g_1 \in H_1$ and $g_2 \in H_2$. Thus, $\ker \phi = H_1 \times H_2$, and we are done by the first homomorphism theorem.

Problem 5. How many necklaces can be made with eight beads of r different colors, if any number of beads of each color can be used? Describe all group actions you are using for counting.

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Solution: This is done similar to Example 5.2.4. Define an action of the dihedral group D_8 on the set X of colorings of beads on a circular wire with a knot not yet tied up. This set has r^8 elements, because we have r choices for each bead independently. To define the action of D_8 on X, place the beads at the vertices of a regular octagon and move the beads around by its symmetries. Note that the number of necklaces is the number of orbits of this action. Now apply Burnside's lemma to this action, for which we need to compute the number of elements in the set $\text{Fix}(g)$ for each $g\ inD_8$. For $\text{Fix}(e)$ is always all X, so $|\text{Fix}(e)| = r^8$. Let R denote counterclockwise rotation of the octagon by $2\pi/8$. Since $g = R$, R^3, R^5, and R^7 generate the same subgroup of D_8, their sets $\text{Fix}(g)$ will be the same; and if a coloring is in $\text{Fix}(R)$, all the beads must be of the same color, and there are r such colorings. Since $g = R^2$ and R^6 generate the same subgroup of D_8, their sets $\text{Fix}(g)$ will be the same; and if a coloring is in $\text{Fix}(R^2)$, every other bead around the circle should be of the same color, and there are r^2 such colorings. The set $\text{Fix}(R^4)$ will have every opposite bead on the other side of the circle colored the same; thus, there are r^4 colorings like that. For each of the four flips a over axes passing through the opposite vertices, the sets $\text{Fix}(a)$ will have vertices across the axis colored the same. There are three such pairs, which gives r^3 colorings; whereas the two vertices on the axis may be colored in any way, which results in a total of r^5 colorings. Finally, for each of the four flips b in axes passing through the midpoints of the opposite sides, the sets $\text{Fix}(b)$ will have vertices across the axis colored the same; resulting in r^4 colorings. Collecting all these in Burnside's formulas, we get

$$\frac{1}{16}(r^8 + 4r + 2r^2 + r^4 + 4r^5 + 4r^4)$$

orbits and thereby necklaces.

Problem 6. Find all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$.

Solution: If $\phi : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism, it will satisfy $\phi(a+b) = \phi(a)+\phi(b)$ and thus, $\phi(m) = m\phi(1) = mn$, where n denotes $\phi(1)$. Now, since we also have $\phi(ab) = \phi(a)\phi(b)$, we get $(ab)n = (an)(bn)$ for each $a, b \in \mathbb{Z}$, in particular, for $a = b = 1$: $n = n^2$, which yields $n = 0$ or 1. Thus, every homomorphism must be either $\phi(m) = 0$ or $\phi(m) = m$, and it is clear that both maps are homomorphisms. Thus, this describes all of them.

Problem 7. Let I be the set of all polynomials in $\mathbb{Z}[x]$ that have an even number as the constant term. Prove that I is an ideal of $\mathbb{Z}[x]$. Is it principal? Is it maximal?

Solution: To see that I is an ideal, we check that the sum of two polynomials with even constant term is a polynomial like that and the negative of such polynomial is a polynomial with even constant term; also, if we multiply such a polynomial by any other polynomial, we will also get an even constant term in the result. The ideal I is not principal, because the two polynomials 2 and $x + 2$ in I do not have common factors in $\mathbb{Z}[x]$. The ideal I is maximal, because $\mathbb{Z}[x]/I \cong \mathbb{Z}_2$ (the isomorphism comes from a homomorphism $p(x) \mapsto p(0) \pmod 2$), which is a field.