Math 127a: Advanced Calculus
Final Exam (Fall 2002)

Write your name and student ID number in the upper right-hand corner of this sheet and write your initials on each page of your exam.

Each problem is worth the same number of points (except for the extra credit). You **must** justify or defend your answer for each problem.

1. Recall that if $A \subset [-\infty, \infty]$, the least upper bound of A is denoted by $\sup A$. Pick the correct word: $\sup A \cap B$ is $\{$always, sometimes, never$\}$ the minimum of $\sup A$ and $\sup B$.

2. Prove that if $a, b \in \mathbb{R}$, then

$$\sqrt{a^2 + b^2} \leq \max\{|a|, |b|\}.$$

3. Find all of the sublimits (limits of convergent subsequences) of this sequence:

$$0.3, 0.23, 0.123, 0.3123, 0.23123, 0.123123, 0.3123123, \ldots$$

4. Give an example of two convergent sequences (s_n) and (t_n) in $\mathbb{R}$, such that the sequence $(\frac{s_n}{t_n})$ is defined but does not converge. (That is, the limit does not exist at all; it's not even ∞ or $-\infty$.)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1$ when x is rational and $f(x) = x$ when x is irrational. Draw f, and prove that f is continuous at 1. (You can use either ϵ and δ continuity or sequential continuity, as you prefer.)

6. This question is about uniform continuity. Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. let $\epsilon = \frac{1}{2}$. Find a δ such that if $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

7. True or false: If $f, g, h : \mathbb{R} \to \mathbb{R}$ be three functions. If f and g are both differentiable at x, if $f'(x) = g'(x)$, and if $f \leq h \leq g$, then h is also differentiable at x and $h'(x) = f'(x)$.

8. Prove that if $f : [a, b] \to \mathbb{R}$ is differentiable at a and has a maximum at a, then $f'(a) \leq 0$.

X. (Extra 1/2 credit) Differentiability is defined just as well for a complex-valued function $f : [a, b] \to \mathbb{C}$ as for a real-valued function. Suppose that such an f is differentiable. Does the mean-value theorem hold?