Name:____

Instructions. *Show all work in the space provided. Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading*. No books or notes are allowed. A scientific calculator is ok - but not needed . The *maximum total score* is 200.

Part I - 96 points. Answer 12 of the following 18 questions. **Circle** the numbers of the 12 questions you want counted - *no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk free!*

1. For each statement, either state that it is *True* or else *Give a Counterexample*:

   (a) If $a < b$ and $c < d$ then $a - c < b - d$.

   (b) If $a < b$ and $c < d$ then $a + c < b + d$.

2. Give an example of an *un*bounded sequence $x_n \in \mathbb{R}$ for which $|x_n| \nrightarrow \infty$.

3. True or False: The sequence x_n, described as follows, is a Cauchy sequence: $0, 1, \frac{3}{2}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{13}{4}, \frac{14}{4}, \frac{15}{4}, 4, \ldots$. (Here the steps between $n \in \mathbb{N} \cup \{0\}$ and $n + 1$ are of size $\frac{1}{n+1}$.)

4. Let $x_n = (-1)^n + \frac{1}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

5. Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Find an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E which has *no* finite subcover

6. Give an example of an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of the interval $(0, \infty)$ for which $\mathcal{O}$ contains *no* finite subcover of $(0, \infty)$.

7. Does the sequence $f_n(x) = x^n$ *converge uniformly*, *converge only point-wise*, or *fail to converge even point-wise* on each of the following intervals?

   (a) $[0, 2]$

   (b) $[0, \frac{1}{2}]$

   (c) $[0, 1)$

8. Let $E \subseteq \mathbb{R}$. True or False:

   (a) E is the union of some family of *closed* sets.

   (b) E is the union of some family of *open* sets.

9. Give an example of a function f which has the *intermediate value property* on $[0, 1]$ yet $f \notin \mathcal{C}[0, 1]$.

10. Give an example of a function $f \in \mathcal{C}[0, 1]$ for which there is *no* point $c \in [0, 1]$ such that $f(c) = c$.

11. Give an example of a function f which is uniformly continuous on the interval $[0, n]$ for *each* $n \in \mathbb{N}$, yet f is not uniformly continuous on $[0, \infty)$.

12. Find $\|f\|_{\sup}$ if $f(x) = \begin{cases} 1 - x & \text{if } x \in [-1, 1] \setminus \mathbb{Q} \\ x^2 & \text{if } x \in [-1, 1] \cap \mathbb{Q}. \end{cases}$

13. Let $T : \mathcal{C}[0, 1] \to \mathbb{R}$ be a bounded linear functional defined by $T(f) = \int_0^1 f(x)(1 + x^2)dx$. Find a real constant K for which $|T(f)| \leq K\|f\|_{\sup}$ for all $f \in \mathcal{C}[0, 1]$.

14. True or Give a Counterexample: If $f \in \mathcal{R}[a,b]$ and if $F(x) = \int_a^x f(t)\ dt$ for all $x \in [a,b]$, then $F'(x)$ exists and equals $f(x)$ for all $x \in [a,b]$.

15. Let $f(x) = \sin x$ for all $x \in [0, 2\pi]$. Find $\int_0^{2\pi} f^-(x)\ dx$.

16. Let $f = 1_{\mathbb{Q}\cap[a,b]}$, the indicator function of the set of all rational numbers. Find the numerical values of $\overline{\int}_a^b f$ and $\underline{\int}_a^b f$.

17. Find $\int_0^1 g(x)\ dx$ if

$$g(x) = \begin{cases} 2x\cos\frac{\pi}{x} + \pi\sin\frac{\pi}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

18. Let $q(x) = \sqrt{x}$ for all $x \in [0,1]$. True or False:

    (a) The function q is uniformly continuous on $(0,1)$.

    (b) The function q' is bounded on $(0,1)$.

Part II - 104 points. Prove carefully *four* of the following *six* theorems. *Circle* the letters of the **four** proofs to be counted - *no more than four!* You may write the proofs below, on the back, or on scratch paper.

A. Suppose A and B are subsets of $\mathbb{R}$, both non-empty, with the special property that $a \le b$ for all $a \in A$ and for all $b \in B$. Prove:

$$\sup(A) \le \inf(B)$$

(Hint: Every b is an upper bound of A. So how does the $\sup(A)$ relate to each $b \in B$?)

B. Suppose that the sequence x_n of real numbers has no convergent subsequences. Let $M > 0$. Prove that there exist at most finitely many values of n such that $x_n \in [-M, M]$. Explain why this implies $|x_n| \to \infty$ as $n \to \infty$.

C. Let $p(x) = a_{2n}x^{2n} + \cdots + a_1 x + a_0$ be any polynomial of *even degree.* Prove: If $a_{2n} > 0$ then p has a *minimum* value on $\mathbb{R}$.

D. Let $f(x) = \begin{cases} 1-x & \text{if } x \in \mathbb{Q} \\ 1-x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Prove that f is continuous at p if and only if $p \in \{0, 1\}$.

E. Suppose $f \in \mathcal{C}(a,b)$ and also that f is bounded on $[a,b]$. Prove that $f \in \mathcal{R}[a,b]$. (Hint: Use a form of the Darboux criterion.)

F. (i) Suppose $|f'(x)| \le M \in \mathbb{R}$ for all $x \in I$, an interval. Prove: f is *uniformly continuous* on I.

(ii) Let $s(x) = \sin \frac{\pi}{x}$ on $(0,1)$. Prove that s is not uniformly continuous on $(0,1)$.

Solutions

Part I

1. For each statement, either state that it is *True* or else *Give a counterexample*:

   (a) If $a < b$ and $c < d$ then $a - c < b - d$.
   Solution: Counterexample: $2 - (-1) > 3 - 1$ although $-1 < 1$ and $2 < 3$.

(b) If $a < b$ and $c < d$ then $a + c < b + d$.
Solution: True.

2. Give an example of an *un*bounded sequence $x_n \in \mathbb{R}$ for which $|x_n| \nrightarrow \infty$.

   Solution: For example, $x_n = (1 + (-1)^n)^n$.

3. True or False: the sequence x_n, described as follows, is a Cauchy sequence: $0, 1, \frac{3}{2}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{13}{4}, \frac{14}{4}, \frac{15}{4}, 4, \ldots$. (Here the steps between $n \in \mathbb{N} \cup \{0\}$ and $n + 1$ are of size $\frac{1}{n+1}$.)

   Solution: False. This sequence cannot be Cauchy since it is not bounded, for which reason it is not convergent.

4. Let $x_n = (-1)^n + \frac{1}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

   Solution: $\limsup x_n = 1$ and $\liminf x_n = -1$.

5. Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Find an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E which has no finite subcover

   Solution: For example, let each $O_n = \left(\frac{1}{2n}, \frac{3}{2n}\right)$.

6. Give an example of an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of the interval $(0, \infty)$ for which $\mathcal{O}$ contains no finite subcover of $(0, \infty)$.

   Solution: For example, let each $O_n = (0, n)$.

7. Does the sequence $f_n(x) = x^n$ *converge uniformly*, *converge only point-wise*, or *fail to converge even point-wise* on each of the following intervals?

   (a) $[0, 2]$
   Solution: f_n fails to converge even point-wise on $[0, 2]$ since it is divergent on $(1, 2]$.

   (b) $[0, \frac{1}{2}]$
   Solution: f_n converges uniformly on $[0, \frac{1}{2}]$ since $\|f_n - 0\|_{\sup} = \frac{1}{2^n} \to 0$.

   (c) $[0, 1)$
   Solution: f_n converges only point-wise to 0 on $[0, 1)$ since

   $$\|f_n - 0\|_{\sup} \equiv 1 \nrightarrow 0$$

   as $n \to \infty$.

8. Let $E \subseteq \mathbb{R}$. True or False:

(a) E is the union of some family of *closed* sets.

Solution: True.

$$E = \bigcup_{e \in E} \{e\}$$

and each singleton set is closed.

(b) E is the union of some family of *open* sets.

Solution: False. E need not be open but the union of every family of open sets is open.

9. Give an example of a function f which has the *intermediate value property* on $[0,1]$ yet $f \notin \mathcal{C}[0,1]$.

Solution: For example, let $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$.

10. Give an example of a function $f \in \mathcal{C}[0,1]$ for which there is no point $c \in [0,1]$ such that $f(c) = c$.

Solution: For example, let $f(x) = 2$ for all $x \in [0,1]$.

11. Give an example of a function f which is uniformly continuous on the interval $[0,n]$ for *each* $n \in \mathbb{N}$, yet f is *not* uniformly continuous on $[0,\infty)$.

Solution: For example, let $f(x) = x^2$ on $[0,\infty)$.

12. Find $\|f\|_{\sup}$ if $f(x) = \begin{cases} 1 - x & \text{if } x \in [-1,1] \setminus \mathbb{Q} \\ x^2 & \text{if } x \in [-1,1] \cap \mathbb{Q}. \end{cases}$

Solution: $\|f\|_{\sup} = 2$. One can see this by considering a sequence of irrational numbers $x_n \to -1$ from the right.

13. Let $T : \mathcal{C}[0,1] \to \mathbb{R}$ be a bounded linear functional defined by $T(f) = \int_0^1 f(x)(1+x^2)dx$. Find a real constant K for which $|T(f)| \leq K\|f\|_{\sup}$ for all $f \in \mathcal{C}[0,1]$.

Solution: One may choose any $K \geq \frac{4}{3}$. This can be seen as follows.

$$|T(f)| = \left| \int_0^1 f(x)(1+x^2)\, dx \right| \leq \int_0^1 |f(x)|(1+x^2)\, dx$$

$$\leq \|f\|_{\sup} \int_0^1 1 + x^2\, dx = \frac{4}{3}\|f\|_{\sup}$$

14. True or Give a Counterexample: If $f \in \mathcal{R}[a,b]$ and if $F(x) = \int_a^x f(t)\ dt$ for all $x \in [a,b]$, then $F'(x)$ exists and equals $f(x)$ for all $x \in [a,b]$.

Solution: Counterexample: Let $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 2. \end{cases}$ Then $F'(1)$ does not exist. One can see this by calculating a simple formula for $F(x)$.

15. Let $f(x) = \sin x$ for all $x \in [0, 2\pi]$. Find $\int_0^{2\pi} f^-(x)\ dx$.

Solution: $\int_0^{2\pi} f^-(x)\ dx = \int_\pi^{2\pi} -\sin x\ dx = \cos 2\pi - \cos \pi = 2$.

16. Let $f = 1_{\mathbb{Q}\cap[a,b]}$, the indicator function of the set of all rational numbers in $[a,b]$. Find the numerical values of $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$.

Solution: $\overline{\int_a^b} f = b - a$ and $\underline{\int_a^b} f = 0$.

17. Find $\int_0^1 g(x)\ dx$ if

$$g(x) = \begin{cases} 2x\cos\frac{\pi}{x} + \pi\sin\frac{\pi}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

Solution: Observe that $g \in \mathcal{R}[0,1]$ since $g \in \mathcal{C}(0,1)$ and g is bounded on $[0,1]$. Note also that $G'(x) \equiv g(x)$ on $[0,1]$ if

$$G(x) = \begin{cases} x^2\cos\frac{\pi}{x} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Thus $\int_0^1 g(x)\ dx = G(1) - G(0) = -1$.

18. Let $q(x) = \sqrt{x}$ for all $x \in [0,1]$. True or False:

(a) The function q is uniformly continuous on $(0,1)$.
Solution: True, since $q \in \mathcal{C}[0,1]$.

(b) The function q' is bounded on $(0,1)$.
Solution: False. Just calculate $q'(x)$ explicitly.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100(A)	11	3	6	3	4
80-89 (B)	4	4	5	1	8
70-79 (C)	2	9	5	7	5
60-69 (D)	0	3	1	6	2
0-59 (F)	3	0	1	2	0
Test Avg	84.6%	78.3%	78.6%	72.6%	78.1%
HW Avg	3.85	5	4.5	4.44	4.44
Test + HW Avg	88.5	83.3	83.1	77	82.5