Math 4606. Spring 2007

Solutions to Homework 10

Problem 7.3.3. $\sum_0^\infty a_n x^{kn}$ converges when $|x^k| < R$ and diverges when $|x^k| > R$, so the radius of convergence is $R^{1/k}$.

Problem 7.3.5a. $e^{-t^2} = \sum_0^\infty (-t^2)^n/n!$ for $t \in \mathbb{R}$. By Theorem 7.18, $\int_0^x e^{-t^2} dt = \sum_0^\infty (-1)^n x^{2n+1}/n!(2n+1)$ for $x \in \mathbb{R}$.

Problem 7.3.7. If $f(x) = f(-x)$, then $\sum a_n x^n = \sum (-1)^n a_n x^n$, so by Corollary 7.22, $a_n = (-1)^n a_{-n}$ for all n and hence $a_n = 0$ for n even.

Problem 8.1.2. By the double angle formula, $\sin^2\theta = (1 - \cos 2\theta)/2$, and the right-hand side is the Fourier series. The reason is as follows. If a function $f(\theta)$ is already represented as a finite sum of $\cos n\theta$ and $\sin n\theta$, say, $A_0/2 + \sum_{n=1}^k (A_n \cos n\theta + B_n \sin n\theta)$, then the Fourier coefficients a_n and b_n can be read off as $a_n = B_n$ and $b_n = B_n$ for all n. To see that, you can rewrite the trigonometric sum as a finite sum $f(\theta) = \sum_{n=-k}^k C_n e^{in\theta}$ by solving the system of equations

$$A_n = C_n + C_{-n}, \qquad B_n = i(C_n - C_{-n}), \tag{1}$$

for C_n and C_{-n}. Then for each $m \in \mathbb{Z}$, the Fourier coefficient c_m for $f(\theta)$ will be given by the following computation:

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-im\theta}d\theta = \frac{1}{2\pi}\sum_{n=-k}^{k} C_n \int_{-\pi}^{\pi} e^{i(n-m)\theta}d\theta = C_m.$$

Thus, the Fourier coefficients c_n may be read off from $\sum C_n e^{in\theta}$ as $c_n = C_n$. Since the trigonometric Fourier coefficients a_n and b_n are obtained from c_n's by Equations (1) rewritten for lower-case letters, it implies that $a_n = A_n$ and $b_n = B_n$ for all n.

Problem 8.1.3.

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m\theta}{4m^2 - 1}$$

Problem 8.1.4. f is even, so $b_n = 0$ and $a_n = (2/\pi)\int_0^\pi \theta^2 \cos n\theta d\theta$. The constant term is $a_0/2 = (1/\pi)\int_0^\pi \theta^2 d\theta = \pi^2/3$. For $n > 0$, integration by parts gives $\int \theta^2 \cos n\theta d\theta = (2/n^2)\theta\cos n\theta + ((\theta^2/n) - (2/n^3))\sin n\theta$, so $a_n = (2/\pi)(2/n^2)(-1)^n\pi = 4(-1)^n/n^2$.

Problem 8.1.5. Here it is easier to use the exponential form of the series: $c_n = (1/2\pi)\int_{-\pi}^{\pi} e^{b\theta}e^{-in\theta}d\theta = \left[e^{(b-in)\theta}\right]_{-\pi}^{\pi}/2\pi(b-in) = (-1)^n(e^{b\pi} - e^{-b\pi})/2\pi(b - in) = (-1)^n(\sinh b\pi)/\pi(b - in)$.

Problem 8.1.9. Suppose $(k-1)P \le a < kP$ for some integer k. Then $\int_a^{a+P} = \int_a^{kP} + \int_{kP}^{a+P}$ (the integrand is $f(x)dx$ in all integrals). By periodicity of f and changing variables from x to $x - P$, the second integral on the right-hand side equals $\int_{(k-1)P}^{a}$, so adding it to the first integral gives $\int_{(k-1)P}^{kP}$. Another application of periodicity combined with a change of variable from x to $x-(k-1)P$ shows that the last integral is equal to $\int_0^P f(x)dx$.