8.2 Convergence of Fourier Series

4. (a) Setting $\theta = 0$ gives $(2/\pi) - (4/\pi)\sum_1^\infty 1/(4m^2 - 1) = 0$ or $\sum_1^\infty 1/(4m^2 - 1) = \frac{1}{2}$, a result also obtainable from the observation that $(4m^2 - 1)^{-1} = \frac{1}{2}[(2m-1)^{-1} - (2m+1)^{-1}]$, so that the series telescopes. Setting $\theta = \frac{1}{2}\pi$ gives $1 = (2/\pi) - (4/\pi)\sum(-1)^m/(4m^2 - 1)$, or $\sum_1^\infty(-1)^{m+1}/(4m^2 - 1) = (\pi - 2)/4$.

   (b) Setting $\theta = \pi$ gives $\pi^2 = (\pi^2/3) + 4\sum_1^\infty 1/n^2$ or $\sum_1^\infty 1/n^2 = \pi^2/6$; setting $\theta = 0$ gives $0 = (\pi^2/3) + 4\sum_1^\infty(-1)^n/n^2$ or $\sum_1^\infty(-1)^{n-1}/n^2 = \pi^2/12$.

   (c) Setting $\theta = 0$ gives $1 = [(\sinh \pi b)/\pi]\sum_{-\infty}^\infty(-1)^n/(b - in)$. The $n = 0$ term is $1/b$, and for $n > 0$ the sum of the nth and $(-n)$th terms is $2b(-1)^n/(b^2 + n^2)$; thus $1 = [(\sinh \pi b)/\pi][(1/b) + 2b\sum_1^\infty(-1)^n/(b^2 + n^2)]$, or $\sum_1^\infty(-1)^n/(b^2 + n^2) = (\pi b \operatorname{csch} \pi b - 1)/2b^2$. Setting $\theta = \pi$ gives $[(\sinh \pi b)/\pi]\sum_{-\infty}^\infty 1/(b - in) = \frac{1}{2}(e^{\pi b} + e^{-\pi b}) = \cosh \pi b$. (The function represented by the series is discontinuous at $\theta = \pi$, so the sum of the series is the average of the left and right hand limits!) Again the $n = 0$ term is $1/b$, and for $n > 0$ the sum of the nth and $(-n)$th terms is $2b/(b^2 + n^2)$, so $(1/b) + \sum_1^\infty 2b/(b^2 + n^2) = \pi \coth \pi b$ and hence $\sum_1^\infty 1/(b^2 + n^2) = (\pi b \coth \pi b - 1)/2b^2$.