8.3 Derivatives, Integrals, and Uniform Convergence

3. (a) From Exercise 4, §8.1, we have $3\theta^2 - \pi^2 = 12\sum_1^\infty[(-1)^n/n^2]\cos n\theta$ ($|\theta| \le \pi$), and termwise integration yields $\theta^3 - \pi^2\theta = 12\sum_1^\infty[(-1)^n/n^3]\sin n\theta$. ($\theta^3 - \pi^2\theta$ is odd, so its mean value on $[-\pi, \pi]$ is 0.)

(b) Integration of the result of (a) and multiplication by 4 gives $\theta^4 - 2\pi^2\theta^2 = C_0 + 48\sum_1^\infty[(-1)^{n+1}/n^4]\cos n\theta$, where $C_0 = (1/2\pi)\int_{-\pi}^{\pi}(\theta^4 - 2\pi^2\theta^2)\,d\theta = (1/\pi)(\frac{1}{5}\pi^5 - \frac{2}{3}\pi^5) = -\frac{7}{15}\pi^4$.

(c) Setting $\theta = \pi$ in (b) gives $-\pi^4 = -\frac{7}{15}\pi^4 - 48\sum_1^\infty(1/n^4)$, or $\sum_1^\infty(1/n^4) = \pi^4/90$.

4. The function $f(\theta) = |\sin\theta|$ is continuous and piecewise smooth, and its derivative is $s(\theta)\cos\theta$ where s is the square wave (Exercise 1, §8.1). So by Corollary 8.27, we have $s(\theta)\cos\theta = (8/\pi)\sum_1^\infty(n\sin 2n\theta)/(4n^2-1)$, and in particular the latter series converges to $\cos\theta$ for $0 < \theta < \pi$. On the other hand, termwise integration of the series for $|\sin\theta|$ from 0 to θ gives, for $0 < \theta < \pi$, $1 - \cos\theta = (2/\pi)\theta - (2/\pi)\sum_1^\infty(\sin 2n\theta)/n(4n^2-1)$, or $\cos\theta = 1 - (2/\pi)\theta + (2/\pi)\sum_1^\infty(\sin 2n\theta)/n(4n^2-1)$. Now, since $4n/(4n^2-1) = (1/n(4n^2-1)) + (1/n)$, the equality of the two series for $\cos\theta$ amounts to the assertion that $\frac{1}{2}\pi - \theta = \sum_1^\infty(\sin 2n\theta)/n$ for $0 < \theta < \pi$, and one verifies this by substituting $\pi - 2\theta$ for θ in Example 1, §8.2.

5. $f(\theta)$ is not continuous (it has jumps at the odd multiples of π), so Theorem 8.26 does not apply. (For those who like distributions: The derivative of $f(\theta)$ is really $f(\theta) - (2\sinh\pi)\delta_\pi(\theta)$, where δ_π is the periodic delta-function with singularities at the odd multiples of π. The Fourier series of δ_π is $(1/2\pi)\sum_{-\infty}^{\infty}(-1)^n e^{in\theta}$, so the correct conclusion is not $c_n = inc_n$ but rather $inc_n = c_n - (-1)^n(\sinh\pi)/\pi$, which is true by Exercise 5, §8.1.)

6. (a) The series $\sum_{n\neq 0} n^{k-(6/5)}/(1+n^6)$ converges if and only if $k \le 6$, so the given series can be differentiated 6 times.

(b) The series $\sum_0^\infty n^k/2^n$ converges for all k, so the given series can be differentiated any number of times.

(c) The given series converges uniformly (M-test with $M_n = 2^{-n}$), so its sum is continuous, but the differentiated series $-\sum\sin 2^n\theta$ does not converge at most points (the terms do not tend to zero as $n \to \infty$).