8.5 Applications to Differential Equations

1. (a) The Fourier cosine series for $f(x) = x$ on $[0, 100]$ is $50 - (400/\pi^2)\sum_1^\infty(\cos(2m-1)\pi x/100)/(2m-1)^2$, so the solution (8.35) of the heat equation is $u(x,t) = 50 - (400/\pi^2)\sum_1^\infty e^{-(.00011)(2m-1)^2\pi^2 t}(\cos(2m-1)\pi x/100)/(2m-1)^2$.

(b) When $t = 60$, the error in discarding the terms after $m = 2$ is

$$\left|\frac{400}{\pi^2}\sum_3^\infty e^{-(.0066)(2m-1)^2\pi^2}\frac{\cos(2m-1)(\pi x/100)}{(2m-1)^2}\right| \le \frac{400}{\pi^2}e^{-(.0066)5^2\pi^2}\sum_3^\infty\frac{1}{(2m-1)^2}$$
$$\approx \frac{400}{\pi^2}e^{-1.628}(.123) \approx .98.$$

To within this error, $u(x,60) = 50-(400/\pi^2)[e^{-.0066\pi^2}\cos(\pi x/100)+\frac{1}{9}e^{-(.0066)9\pi^2}\cos(3\pi x/100)] \approx 50-(37.97)\cos(\pi x/100)-(2.51)\cos(3\pi x/100)$, which is about 10 when $x = 0$, 12 when $x = 10$, and 40 when $x = 40$.

(c) For $t \ge 3600$, $|u(x,t)-50| \le (400/\pi^2)e^{-(.0011)\pi^2(3600)}\sum_1^\infty 1/(2m-1)^2 = 50e^{-(.396)\pi^2} \approx 1.0037$. Almost good enough, but not quite! A slightly less crude estimate works: $|u(x,t)-50| \le (400/\pi^2)[e^{-(.396)\pi^2}+e^{-9(.396)\pi^2}\sum_2^\infty 1/(2m-1)^2] = (400/\pi^2)[e^{-(.396)\pi^2}+e^{-(3.564)\pi^2}((\pi^2/8)-1)] \approx .81$.

2. One follows the separation-of variables procedure as on p. 382 to find solutions of the form $e^{-k\alpha t}(C_1\cos\sqrt{\alpha}\,\theta + C_2\sin\sqrt{\alpha}\,\theta)$. The periodicity condition then forces $\sqrt{\alpha} = 2n\pi$, so the resulting analog of (8.35) is $u(\theta,t) = \sum_0^\infty e^{-4n^2\pi^2kt}(a_n\cos n\theta + b_n\sin n\theta)$. To satisfy the initial condition one takes $\sum_0^\infty(a_n\cos n\theta + b_n\sin n\theta)$ to be the Fourier series of $f(\theta)$. (The result looks a little neater in exponential form: $u(\theta,t) = \sum_{-\infty}^\infty c_n e^{-4n^2\pi^2kt+in\theta}$ where $f(\theta) = \sum_{-\infty}^\infty c_n e^{in\theta}$.)

3. If $u(x,t) = \sum_1^\infty b_n(t)\sin(n\pi x/l)$ is to satisfy $\partial_t u = k\partial_x^2 u + G$ where $G(x,t) = \sum_1^\infty \beta_n(t)\sin(n\pi x/l)$, we must have $b_n'(t) = -k(n\pi/l)^2 b_n(t) + \beta_n(t)$, assuming that termwise differentiation of the series is justified. To solve this ordinary differential equation, multiply through by the integrating factor $e^{k(n\pi/l)^2t}$ to obtain $(d/dt)[b_n(t)e^{k(n\pi/l)^2t}] = e^{k(n\pi/l)^2t}\beta_n(t)$, whence $b_n(t)e^{k(n\pi/l)^2t} = b_n(0) + \int_0^t e^{k(n\pi/l)^2s}\beta_n(s)\,ds$. For this to work, the following conditions are (more than) sufficient: (1) f is of class C^1 on $[0,l]$, and $f(0) = f(l) = 0$. (2) $G(x,t)$ is C^2 as a function of $x \in [0,l]$ for each t, $G(0,t) = G(l,t) = 0$, and $G(x,t)$, $\partial_x G(x,t)$, and $\partial_x^2 G(x,t)$ are jointly continuous as functions of $x \in [0,l]$ and $t \ge 0$. The boundary conditions on f and G guarantee that their odd periodic extensions are still at least C^1, and that of $\partial_x^2 G$ is at least piecewise continuous. It follows that the Fourier sine coefficients of f (namely, $b_n(0)$) are absolutely summable, and those of G (namely, $\beta_n(t)$) are continuous in t and satisfy $|\beta_n(t)| \le Cn^{-2}$ for t in any finite interval $[0,T]$. We then have

$$|b_n(t)| \le e^{-k(n\pi/l)^2t}\left[|b_n(0)| + Cn^{-2}\int_0^t e^{-k(n\pi/l)^2s}\,ds\right] \le e^{-k(n\pi/l)^2t}|b_n(0)| + \frac{C}{k(\pi/l)^2n^4}.$$

This is enough to guarantee the absolute and uniform convergence of the series defining $u(x,t)$ for $x \in [0,l]$ and $t \in [0,T]$, as well as the absolute and uniform convergence of the series defining $\partial_t u(x,t)$ and $\partial_x^2 u(x,t)$ for $x \in [0,l]$ and $t \in [\epsilon,T]$ ($\epsilon > 0$), so that all formal calculations are justified.

4. (a) The odd periodic extension of the initial displacement $u(x,0)$ is $mg(\pi x/l)$ where g is as in Exercise 2, §8.3, with $a = \pi b/l$, so its Fourier sine series can be read off from the answer to that exercise. The series for $u(x,t)$ can then be read off from (8.37).

(b) When $b = (0.4)l$ we have $2l^2/\pi^2b(l-b) = 200/24\pi^2 \approx .844$, and $n^{-2}\sin((.4)n\pi) \approx .951, .147, -.065, -.059, 0$ when $n = 1,2,3,4,5$, so the first five coefficients (up to the overall factor of m) are $.803, .124, -.055, -.050, 0$. When $b = (0.1)l$ we have $2l^2/\pi^2b(l-b) = 200/9\pi^2 \approx 2.252$, and $n^{-2}\sin((.1)n\pi) \approx .309, .147, .090, .059, .040$ when $n = 1,2,3,4,5$, so the first five coefficients are (m times) $.696, .331, .203, .133, .090$. (Note: The L^2 norm of the initial displacement $u(\cdot,0)$ is $m\sqrt{l/3}$, independent of b, so the total energy of these waves is independent of b and a direct comparision of the coefficients is appropriate.)