Math 4606. Spring 2007

Solutions to Homework 4.5

Problem 1.7.3. If f is neither strictly increasing, nor strictly decreasing, one can find points $x, y, z \in I$ such that (i) $x < y < z$, and (ii) either $f(x) \leq f(y)$ and $f(y) \geq f(z)$, or $f(x) \geq f(y)$ and $f(y) \leq f(z)$; we assume the former alternative. The latter is done exactly the same way. If $f(x) = f(y)$ or $f(y) = f(z)$, then f is not one-to one. Otherwise, the intervals $(f(x), f(y))$ and $(f(z), f(y))$ are nonempty, and one is contained in the other.Assuming f is continuous, the intermediate value theorem implies that the image $f((x, y))$ of the interval (x, y) contains the interval $(f(x), f(y))$ and $f((y, z)) \supset (f(z), f(y))$, so there are points in (x, y) and (y, z) at which f takes the same value, and again f is not one-to-one.

Problem 1.7.4. Suppose $S_1 \cup S_2$ is disconnected, so $S_1 \cup S_2 = U \cup V$, where neither U, nor V intersects the closure of the other one. Then $S_1 = (S_1 \cap U) \cap (S_1 \cap V)$ is a disconnections of S_1, unless either $S_1 \cap V$ or $S_1 \cap U$ is empty, i.e., $S_1 \subset U$ or $S_1 \subset V$. Likewise, we must have $S_2 \subset U$ or $S_2 \subset V$. It cannot be that S_1 and S_2 are both contained in U (resp., V), for then V (resp., U) would be empty so $S_1 \subset U$ and $S_2 \subset V$ or vice versa. Either alternative contradicts the assumption that $S_1 \cap S_2 \neq \emptyset$.

$S_1 \cap S_2$ is connected when $n = 1$ by Theorem 1.25, but not when $n > 1$. For example, take S_1 to be the unit sphere (Exercise 2) and S_2 to be a line through the origin; the intersection consists of two points.

Problem 1.7.10. $f(1, 3) = -2$ and $f(4, -1) = 5$, so there is a point $(x, y) \in S$ such that $f(x, y) = 0$, i.e., $x = y$.