Math 4606. Spring 2007

Solutions to Homework 8

Problem 2.5.6. $\partial x/\partial y$ is the derivative when x is considered as a function of y and z; by (2.44) from the text, it equals $-\partial_y F/\partial_x F$. Likewise, $\partial y/\partial z = -\partial_z F/\partial_y F$ and $\partial z/\partial x = -\partial_x F/\partial_z F$. The product of these quantities is -1.

Problem 2.6.9. In one variable, the assertion is that $(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}$, which is proven just like the binomial theorem. (Induction on k, using the fact that $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$.) The n-variable result follows by applying the one-variable result in each variable separately; the facts that $\alpha = \beta + \gamma$ is equivalent to $\alpha_j = \beta_j + \gamma_j$ for all j and $\alpha! = \alpha_1! \ldots \alpha_n!$ make everything turn out right. This could be phrased as an induction on n.

Problem 2.7.2. (a) $f'(x) = x^{-1}, f''(x) = -x^{-2}$, and $f^{(3)}(x) = 2x^{-3}$, so $P_{1,3}(h) = h - h^2/2 + h^3/3$; also $|f^{(4)}(x)| = |-6x^{-4}| \leq 96$ for $|x-1| \leq 1/2$, so $C = 96/4! = 4$.
(b) $f'(x) = x^{-1/2}/2$, $f''(x) = -x^{-3/2}/4$, and $f^{(3)}(x) = 3x^{-5/2}/8$, so $P_{1,3}(h) = 1 + h/2 - h^2/8 + h^3/16$; also $|f^{(4)}(x)| = |-15x^{-7/2}/16| \leq \frac{15}{16} 2^{7/2}$ for $|x-1| \leq 1/2$, so $C = \frac{15}{16} 2^{7/2}/4! = 5 \cdot 2^{-7/2}$.
(c) $f'(x) = -(x+3)^{-2}$, $f''(x) = 2(x+3)^{-3}$, and $f^{(3)}(x) = -6(x+3)^{-4}$, so $P_{1,3}(h) = 1/4 - h/16 + h^2/64 - h^3/256$; also $|f^{(4)}(x)| = |24(x+3)^{-5}| \leq 24(3.5)^{-5}$ for $|x-1| \leq 1/2$, so $C = (3.5)^{-5}$.

Problem 2.7.7. With $x = 1+h$, $y = 2+k$, $z = 1+l$, we have $f(x,y,z) = (1+h)^2(2+k) + (1+l) = 3 + 4h + k + l + 2h^2 + 2hk + h^2 k$, with no remainder. Since this is a third-degree polynomial, by the uniqueness theorem of Taylor polynomials, Theorem 2.77, this must be the 3rd-order Taylor polynomial for the given function. The remainder is zero, because the function equals its 3rd Taylor polynomial precisely. You may also show that the remainder vanishes, because all 4th-order partial derivatives of f vanish: the 3rd-order partials of f must be polynomials of degree zero, i.e., constants, and their derivatives will be equal to zero.

Problem 2.7.8. A $(k-1)$-fold application of l'Hôpital gives

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+h) - P_{a,k}(h)}{h^k} &= \lim_{h\to 0} \frac{f^{(k-1)}(a+h) - f^{(k-1)}(a) - f^{(k)}(a)h}{h} \\ &= \lim_{h\to 0} \frac{f^{(k-1)}(a+h) - f^{(k-1)}(a)}{h} - f^{(k)}(a) = 0\end{aligned}$$

by definition of $f^{(k)}(a)$.

Problem 2.7.9. We have $f(a+h) = f(a) + f^{(k)}(a)h^k/k! + R_{a,k}(h)$, and by Corollary 2.60, for h sufficiently small, we have $|R_{a,k}(h)| \leq \frac{1}{2}|f^{(k)}(a)h^k|/k!$.

Thus for k even, if $f^{(k)}(a) > 0$, we have $f(a+h) - f(a) < 0$ for small h. For k odd, the same reasoning shows that $f(a+h) - f(a)$ changes sign along with h^k at $h = 0$.