Math 4606. Spring 2007

Solutions to Homework 9.5

Problem 3.1.3. With $u = (x^2 + y^2 + 2z^2)^{1/2}$ and $F(x, y, z) = u - \cos z$, we have $F_y = y/u$ and $F_z = 2z/u + \sin z$, so $F_y(0, 1, 0) = 1$ and $F_z(0, 1, 0) = 0$. Hence the equation can be solved for y, but not z.

Problem 3.1.5. With $G(x, y) = F(F(x, y), y)$, we have $G_y = F_1(F(x, y), y)F_2(x, y) + F_2(F(x, y), y)$, so $G_y(0, 0) = F_2(0, 0)(F_1(0, 0) + 1) \neq 0$ when $F_2(0, 0) \neq 0$ and $F_1(0, 0) \neq -1$.

Problem 3.1.7. With $(z, w) = \mathbf{F}(x, y, u, v) = (u^3 + xv - y, v^3 + yu - x)$, we have $D\mathbf{F} = \begin{pmatrix} v & -1 & 3u^2 & x \\ -1 & u & y & 3v^2 \end{pmatrix}$. At $(0, 1, 1, -1)$, then, we have $D\mathbf{F} = \begin{pmatrix} -1 & -1 & 3 & 0 \\ -1 & 1 & 1 & 3 \end{pmatrix}$. The determinants of all 2×2 submatrices of this matrix are nonzero, so the equations can be solved for any pair of variables.

Problem 3.1.9. With $u = x^2 + y^2 + z^2$, $v = xy + tz$, and $w = xz + ty + e^t$, we have $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det \begin{pmatrix} 2x & 2y & 2z \\ y & x & t \\ z & t & x \end{pmatrix}$, which equals 8 at $(x, y, z, t) = (-1, -2, 1, 0)$. So the equations can be solved for x, y, z.

Problem 7.1.1. (a) $\lim_{k\to\infty} f_k(x) = 0$, if $0 \leq x < 1$, while $\lim_{k\to\infty} f_k(1) = 1$. We have $|f_k(x) - 0| \leq (1 - \delta)^k \to 0$ for $x \in [0, 1 - \delta]$, so the convergence is uniform there.

(b) $\lim_{k\to\infty} f_k(0) = 0$, while $\lim_{k\to\infty} f_k(x) = 1$, if $0 < x \leq 1$. We have $|f_k(x) - 1| = 1 - \delta^{1/k} \to 0$ for $x \in [\delta, 1]$, so the convergence is uniform there.

(c) $\lim_{k\to\infty} f_k(x) = 0$, if $x \in [0, \pi] \setminus \{\pi/2\}$, while $\lim_{k\to\infty} f_k(\pi/2) = 1$. We have $|f_k(x) - 0| \leq \sin^k(\pi/2 - \delta) \to 0$ for $x \in [0, \pi/2 - \delta]$ or $x \in [\pi/2 + \delta, 1]$, so the convergence is uniform there.

(d) $|f_k(x)| \leq 1/k$ for all x, so $f_k \to 0$ uniformly on $\mathbb{R}$.

(e) $\lim_{k\to\infty} f_k(x) = 0$ for all $x \in [0, \infty)$, but the maximum of f_k on this interval is e^{-1} (at $x = k^{-1}$), so the convergence is not uniform. However, $|f_k(x) - 0| \leq k\delta e^{-k\delta}$ for $x \geq \delta$, provided $k \geq \delta^{-1}$, and $\lim_{k\to\infty} k\delta e^{-k\delta} = 0$; hence the convergence is uniform on $[\delta, \infty)$.

(f) $\lim_{k\to\infty} f_k(x) = 0$ for all $x \in [0, \infty)$, but the maximum of f_k on this interval is e^{-1} (at $x = k$), so the convergence is not uniform. However, $|f_k(x) - 0| \leq b/k$ for $x \leq b$, so the convergence is uniform on $[0, b]$ for any b.

(g) $\lim_{k\to\infty} f_k(x) = 0$ for all $x \neq 1$, since $f_k(x) < x^k$ for $x < 1$ and $f_k(x) < x^{-k}$ for $x > 1$, and $\lim_{k\to\infty} f_k(1) = 1/2$. For any $\delta > 0$ we have $|f_k(x) - 0| \leq$

$(1-\delta)^k$ for $x \in [0, 1-\delta]$ and $|f_k(x) - 0| \leq (1+\delta)^{-k}$ for $x \in [1+\delta, \infty)$, so the convergence is uniform on these intervals.

Problem 7.1.2. (a) The series is a geometric series, convergent for $x > 0$ to the sum $1/(1-e^{-x})$. The convergence is absolute and uniform on $[\delta, \infty)$ for any $\delta > 0$, by the M-test with $M_n = e^{-n\delta}$. The sum is continuous on $(0, \infty)$.

(b) The series is absolutely and uniformly convergent on $[-1, 1]$ by the M-test with $M_n = 1/n^2$ for $n > 0$; it diverges elsewhere since the nth term $\not\to 0$. The sum is continuous on $[-1, 1]$.

(c) The series is absolutely and uniformly convergent on $[-2+\delta, 2-\delta]$ for any $\delta > 0$ by the M-test with $M_n = n(1-\delta/2)^n/8$, while $\sum_n M_n$ converges by the ratio test. It diverges for $|x| \geq 2$ since the nth term $\not\to 0$. The sum is continuous on $(-2, 2)$.

(d) The series is absolutely and uniformly convergent on $\mathbb{R}$ by the M-test with $M_n = 1/n^3$; the sum is everywhere continuous.

(e) The series is absolutely and uniformly convergent on $\mathbb{R}$ by the M-test with $M_n = 1/n^2$; the sum is everywhere continuous.

(f) The series is absolutely and uniformly convergent on $[1+\delta, \infty)$ for any $\delta > 0$ by the M-test with $M_n = n^{-1-\delta}$, and it diverges for $x \leq 1$ (Theorem 6.9). The sum is continuous on $(1, \infty)$.

Problem 7.1.3. Let M be the maximum value of $|g(x)|$ on $[0, 1]$. Given $\varepsilon > 0$, choose $\delta.0$ so that $|g(x)| < \varepsilon$ for $1 - \delta \leq x \leq 1$. Then if k is large enough so that $(1-\delta)^k < \varepsilon/m$, we have $|f_k(x)| \leq Mx^k < \varepsilon$ for $x \leq 1-\delta$ and $|f_k(x)| \leq |g(x)| < \varepsilon$ for $x > 1-\delta$. That is, $|f_k(x)| < \varepsilon$ for all $x \in [0, 1]$, when k is sufficiently large, so $f_k \to 0$ uniformly on $[0, 1]$.

Problem 7.1.4. Given $\delta > 0$. let $I_1 = [-1+\delta, 1-\delta]$, and for $k \geq 2$ let $I_k = [k-1+\delta, k-\delta]$. For a given k, let $M_n = \max_{x \in I_k} |x^2 - n^2|^{-1}$. Then $M_n < \infty$ for all n, and $M_n/n^{-2} \to 1$ as $n \to \infty$, so $\sum M_n < \infty$. The M-test therefore gives uniform convergence of the series for $x \in I_k$ or $-x \in I_k$.

Problem 7.1.5. The series fails to converge absolutely by comparison to $\sum 1/n$. However, $1/(x^2+n)$ decreases to 0 as $n \to \infty$ for each x, so by the alternating series test, the series converges for each x, and the absolute divergence between the kth partial sum and the full sum is at most $1/(x^2+k+1) \leq 1/(k+1)$. The latter quantity is independent of x and tends to zero as $k \to \infty$, so the convergence is uniform.