Math 4606. Spring 2007

Solutions to Homework 9

Problem 2.9.5. Clearly $|f(x,y)| \geq |(x,y)|^2$, so f has a minimum on $\mathbb{R}^2$ by Theorem 2.83a. We have $f_x = -2b(A - bx - cy) + 2x$ and $f_y = -2c(A - bx - cy) + 2y$. Setting these equal to zero simultaneously gives $x = Ab/(1 + b^2 + c^2)$ and $y = Ac/(1 + b^2 + c^2)$. Substituting these values into $f(x,y)$ and simplifying gives $f = A^2/(1 + b^2 + c^2)$.

Problem 2.9.6. Clearly $f(x,y) > 0$ except when $x = y = 0$, so $f(0,0) = 0$ is the absolute minimum. f has an absolute maximum by Theorem 2.83b, namely $f(0, \pm 1) = 2/e$.

Problem 2.9.9. Lagrange's method works easily here: one has to solve $2x = 2\lambda x$, $4y = 2\lambda y$, and $6z = 2\lambda z$ subject to the constraint $x^2 + y^2 + z^2 = 1$. The first equation is equivalent to $\lambda = 1$ or $x = 0$. If $\lambda = 1$, the second and third equations imply that $y = z = 0$ and hence $x = \pm 1$. If $x = 0$, the second equation forces $\lambda = 2$ or $y = 0$. If $\lambda = 2$, then $z = 0$, so $y = \pm 1$; if $y = 0$, then $z = \pm 1$. So the constrained critical points are $\pm(1,0,0)$, $\pm(0,1,0)$, and $\pm(0,0,1)$. Clearly the first pair gives the minimum of 1 and the last pair gives the maximum of 3.

Solving the constraint first loses either the minimum or the maximum, depending on which variable you solve for. This happens because we step into the realm of nondifferentiable functions by solving the constraint: $z = \sqrt{1 - x^2 - y^2}$. So, at a local extremum of the problem, the function $f(x,y) = x^2 + 2y^2 + 3(1 - x^2 - y^2)$ will have a zero gradient or $z = \sqrt{1 - x^2 - y^2}$ will not be differentiable, and the lost solution will lie in that locus $x^2 + y^2 = 1$, where z is not differentiable as a function of x and y.

Problem 2.9.13. Parametrizing the first line by $\mathbf{f}(s) = (1 - s, s, 0)$ and the second one by $\mathbf{g}(t) = (t,t,t)$,we wish to minimize $\phi(s,t) = |\mathbf{f}(s) - \mathbf{g}(t)|^2 = (1 - s - t)^2 + (s - t)^2 + t^2 = 2s^2 + 3t^2 - 2s - 2t + 1$. We have $\phi_s = 4s - 2$ and $\phi_t = 6t - 2$, so the critical point is $s = 1/2$, $t = 1/3$. The point on the first line closest to the second line is $\mathbf{f}(1/2) = (1/2, 1/2, 0)$ (and the minimum distance is $\sqrt{1/6}$).

Problem 2.9.14. We wish to minimize $V = xyz$ subject to the constraint $xy + 2xz + 2yz = A$. Solving the constraint equation for z gives $V = xy(A - xy)/(2x + 2y)$. After a little calculation, we find $V_x = 2y^2(A - x^2 - 2xy)/(2x + 2y)^2$ and $V_y = 2x^2(A - y^2 - 2xy)/(2x + 2y)^2$. Setting these to zero gives $x^2 + 2xy = A = y^2 + 2xy$, so that $x = y = \sqrt{A/3}$; hence $z = \frac{1}{2}\sqrt{A/3}$ and $V = \frac{1}{2}(\sqrt{A/3})^3$.

Problem 2.9.15. We wish to minimize $x^2 + y^2 + z^2$ subject to the constraints

$x + z = 4$ and $3x - y = 6$. By Lagrange's method, we solve $2x = \lambda + 3\mu$, $2y = -\mu$, and $2z = \lambda$ to obtain $2x = 2z - 6y$; solving this simultaneously with the constraint equations gives $x = z = 2$, $y = 0$.