Problem 2. Prove that if f is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^k$, then for any open subset U of $\mathbb{R}^k$, the pre-image

$$V = \{x \in \mathbb{R}^n \; : \; f(x) \in U\}$$

is an open set in $\mathbb{R}^n$.

For every point $\vec{x} \in V$, take $f(\vec{x})$, which will be in U. Since U is open, $\exists$ a open ball $B(r, f(\vec{x}))$ contained in U. Since $f(\vec{x})$ is continuous at $\vec{x}$, for each $\varepsilon > 0$, in particular, $\epsilon = r$, $\exists\ \delta > 0$: $|f(\vec{y}) - f(\vec{x})| < \varepsilon = r$ whenever $|\vec{y} - \vec{x}| < \delta$,

or, equivalently, $f(\vec{y}) \in B(r, f(\vec{x}))$ whenever $\vec{y} \in B(\delta, \vec{x})$.

This shows that $B(\delta, \vec{x}) \subset V$ (because $B(r, f(\vec{x})) \subset U$).