Problem 4. We know that a function continuous on a compact set in $\mathbb{R}$ will assume an absolute maximum value. Now let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Suppose $\lim_{x \to \infty} f(x) = L$ exists and is finite and also suppose $f(a) > L$ for some $a \geq 0$. Show that f assumes an absolute maximum value on $[0, \infty)$. [Hint: The fact that $\lim_{x \to \infty} f(x) = L$ means that for each $\epsilon > 0$ there exists $K > 0$ such that for each $x > K$ we have $|f(x) - L| < \epsilon$. Show that if $x > K$ for some $K > 0$, then $f(x) < f(a)$.]

Let $\varepsilon = f(a) - L$. Then $\exists\ K > 0$:

$\forall\ x > K$ we have $|f(x) - L| < \varepsilon \Rightarrow$

$L - \varepsilon < f(x) < L + \varepsilon = f(a)$. Make sure to take $K \geqslant a$.

~~Thus~~ On $[0, K]$, since it's compact and $f(x)$ is continuous on $[0, K]$, $f(x)$ will assume an absolute maximum, say

$f(b) = M$, $b \in [0, K]$.

Then $M \geq f(a)$ because $a \in [0, K]$.

~~Also note~~ Then $M > f(x)$ for all $x > K$.

Since M is a max, ~~and~~ $M \geq f(x)$ $\forall\ x \in [0, K]$.

Thus, M is an abs. max. on $[0, \infty)$.

Why not check up all your work?