Math 4606. Spring 2007

Solutions to Another Sample Midterm Exam 2

Problem 1. Take $f(x) = \sin\frac{1}{x}$, which is bounded by 1 and continuous on $(0,1)$ as a composition of continuous functions.

We need to show that there exists $\varepsilon > 0$ such that for each $\delta > 0$ there exist $x, y \in (0,1)$ with $|x-y| < \delta$, but $|\sin\frac{1}{x} - \sin\frac{1}{y}| \ge \varepsilon$. Take $\varepsilon = \sqrt{2}/8$. Given $\delta > 0$, take $k \in \mathbb{N}$ so that

$$\frac{1}{\pi k} - \frac{1}{\pi k + \pi/4} = \frac{1}{\pi k(4k+1)} < \delta.$$

Set $x = \frac{1}{\pi k}$, $y = \frac{1}{\pi k + \pi/4}$. They are in $(0,1)$ and $|x-y| < \delta$. However, $|\sin\frac{1}{x} - \sin\frac{1}{y}| = |\cos\frac{1}{c}|\frac{1}{c^2}|x-y|$ by the Mean Value Theorem (MVT) for some c between x and y. Since $\pi k < 1/c < \pi k + \pi/4$, we have $|\cos\frac{1}{c}| > \sqrt{2}/2$ and $1/c^2 > \pi^2 k^2$, whence $|\sin\frac{1}{x} - \sin\frac{1}{y}| = |\cos\frac{1}{c}||x-y|/c^2 > \frac{\sqrt{2}}{2}\frac{\pi^2 k^2}{\pi k(4k+1)} = \frac{\sqrt{2}\pi k}{2(4k+1)} > \frac{\sqrt{2}2k}{16k} = \frac{\sqrt{2}}{8} = \varepsilon$.

This problem is a little tough for an exam. A problem "Give an example of a continuous function on $(0,1)$ that is not uniformly continuous" would be more appropriate. A function like that would be $f(x) = 1/x$. The explanation why it is not uniformly continuous would be similar, but simpler.

Problem 2. If that equation had a solution, it would mean that for some $x \in (0, \pi/2)$ the 6th-degree Taylor polynomial remainder $\frac{\sin^{(7)} c}{7!}x^7 = 0$ for some c strictly between 0 and x and therefore between 0 and $\pi/2$. But $\sin^{(7)} c = -\cos c < 0$ and $x > 0$ for x and c between 0 and $\pi/2$, thus the remainder is always strictly negative.

It was too quick for me to agree that a solution by taking 7 derivatives of both sides of the original equation, suggested by one of the students, was correct. In fact, if we assume the equality of the values of the two functions at some point x, it will not be necessarily true that the derivatives will be equal even at that point. Thus, only the solution that I presented to you in class was correct.

Problem 3. (1) This is implicit differentiation. Assuming x is a function of y and z, differentiate the given equation with respect to z. We get

$$\frac{2x\frac{\partial x}{\partial z} + z}{\sqrt{2x^2 + z^2 - 2}} = -\sin(yx^2)2yx\frac{\partial x}{\partial z}.$$

We need to evaluate this at $y = 0$, $z = 1$, and $x = -1$. We get

$$-2\frac{\partial x}{\partial z} + 1 = 0$$

and $\frac{\partial x}{\partial z} = \frac{1}{2}$.
(2) Differentiating the first given equation in y, we get

$$\frac{2x\frac{dx}{dy} + z\frac{dz}{dy}}{\sqrt{2x^2 + z^2 - 2}} = -\sin(yx^2)(2yx\frac{dx}{dy} + x^2).$$

Evaluating this at $y = 0$, $z = 1$, and $x = -1$, we obtain

$$-2\frac{dx}{dy} + \frac{dz}{dy} = 0.$$

Thus

$$\frac{dz}{dy} = 2\frac{dx}{dy}. \tag{1}$$

Now let us deal similarly with the second given equation.

$$2\frac{dx}{dy} + 1 + 2z\frac{dz}{dy} = \cos y - \frac{dz}{dy}.$$

After evaluation, get

$$2\frac{dx}{dy} + 3\frac{dz}{dy} = 0.$$

Substituting (1) into that, we see that

$$\frac{dx}{dy} = \frac{dz}{dy} = 0$$

at $y = 0$.

Problem 4. Using Lagrange's method, we look at the points where

$$\nabla(x + y + z) = \lambda\nabla(a/x + b/y + c/z - 1),$$

or componentwise

$$\begin{aligned} 1 &= -a\lambda/x^2, \\ 1 &= -b\lambda/y^2, \\ 1 &= -c\lambda/z^2, \end{aligned}$$

subject to the constraint $a/x + b/y + c/z = 1$. Solve the first equation for λ, substitute into the other two equations and solve them for y and z, remembering that $x, y, z > 0$ by the assumptions of the problem:

$$\lambda = -x^2/a, \tag{2}$$

$$y = \sqrt{b/a}x, \tag{3}$$

$$z = \sqrt{c/a}x. \tag{4}$$

Substitute the y and z into the constraint to get

$$x = a + \sqrt{ab} + \sqrt{ac}.$$

This is the only critical point, delivering a minimal value of $x + y + z$, because it goes to ∞ as $|(x, y, z)| \to \infty$ on $\{a/x + b/y + c/z = 1\}$ by Theorem 2.83a. At that point the minimum value will be attained. That value is equal to

$$\begin{aligned} &x + y + z \\ &\quad = (a + \sqrt{ab} + \sqrt{ac}) + (\sqrt{ab} + b + \sqrt{bc}) + (\sqrt{ac} + \sqrt{bc} + c) \\ &\quad = a + b + c + 2\sqrt{ab} + 2\sqrt{ac} + 2\sqrt{bc}. \end{aligned}$$