Math 4606. Fall 2006.
Solutions to Exam 1

1. (20 points) Let X and Y be two non-empty sets and let f be a one-to-one function from X to Y. Let A be a subset of X. Show that $f(X \setminus A)$ is a subset of $Y \setminus f(A)$.

Solution. Let $y \in f(X \setminus A)$, then $y = f(x)$ for some $x \in X \setminus A$. Suppose $y \in f(A)$. Then there is $x' \in A$: $y = f(x')$. Since A and $X \setminus A$ are disjoint, $x \neq x'$. Therefore $y = f(x) = f(x')$ with $x \neq x'$ which contradicts f being one-to-one. Thus $y \notin f(A)$, which means $y \in Y \setminus f(A)$. Hence $f(X \setminus A) \subset Y \setminus f(A)$.

2. (20 points) Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+5y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Does the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? Why? Find the limit if it does.

Solution. Let us look at $\lim_{x\to 0} f(x,0)$ and $\lim_{x\to 0} f(x,x)$. The first limit exists and is equal to 0, because $f(x,0) = 0$ for all x. The second limit may be computed by computing the function $f(x,x)$ for $x \neq 0$ as follows:

$$f(x,x) = \frac{2x^2}{x^2+5x^4} = \frac{2}{1+5x^2}.$$

Hence $\lim_{x\to 0} f(x,x) = 2$. Thus, we obtain two different limits as (x,y) aproaches $(0,0)$ along two different lines, which implies that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

3. (20 points) Let f, g and h be three real-valued functions on $\mathbb{R}^n$ satisfying

$$g(x) \leq f(x) \leq h(x) \text{ for all } x \in \mathbb{R}^n.$$

Let $a \in \mathbb{R}^n$ and $L \in \mathbb{R}$ and suppose that

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Prove that $\lim_{x\to a} f(x) = L$.

Solution. Let $\varepsilon > 0$. Since $\lim_{x\to a} g(x) = L$, there is $\delta_1 > 0$ such that

$$-\varepsilon < g(x) - L < \varepsilon \text{ whenever } 0 < |x-a| < \delta_1. \tag{1}$$

Similarly, there is $\delta_2 > 0$ such that

$$-\varepsilon < h(x) - L < \varepsilon \text{ whenever } 0 < |x-a| < \delta_2. \tag{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Let x be in $\mathbb{R}^n$ such that $0 < |x - a| < \delta$. We have $0 < |x - a| < \delta_1$ and hence by (1):

$$f(x) - L \geq g(x) - L > -\varepsilon.$$

We have $0 < |x - a| < \delta_2$ and hence by (2):

$$f(x) - L \leq h(x) - L < \varepsilon.$$

Thefore $|f(x) - L| < \varepsilon$. Thus $\lim_{x\to a} f(x) = L$.

4. (20 points) Show that the set

$$S = \{(x, y) \in \mathbb{R}^2 : \ xy > 5 \text{ and } y + x^2 + 3x < 13\}$$

is an open set in $\mathbb{R}^2$.

Solution. We write $S = A \cap B$ where

$$A = \{(x, y) \in \mathbb{R}^2 : \ xy > 5\}, \quad B = \{(x, y) \in \mathbb{R}^2 : y + x^2 + 3x < 13\}.$$

Let $f_1(x, y) = xy$ and $f_2(x, y) = y + x^2 + 3x$. We know that f_1 and f_2 are continuous on $\mathbb{R}^2$. Since $A = f_1^{-1}((5, \infty))$ and $(5, \infty)$ is open, we obtain A is open. Similarly, $B = f_2^{-1}((-\infty, 13))$ is open. Therefore S is open, for being an intersection of two open sets.

5. (20 points) Find the limit

$$\lim_{k\to\infty} \frac{-3k^3 + 8k^2 - 7k + 11}{4k^3 - k^2 + 5}.$$

Solution. Divide the denominator and numerator by k^3, we have

$$\frac{-3k^3 + 8k^2 - 7k + 11}{4k^3 - k^2 + 5} = \frac{a_k}{b_k},$$

where

$$a_k = -3 + \frac{8}{k} - \frac{7}{k^2} + \frac{11}{k^3}, \quad b_k = 4 - \frac{1}{k} + \frac{5}{k^3}.$$

We have $\lim_{k\to\infty} a_k = -3$ and $\lim_{k\to\infty} b_k = 4 \neq 0$, hence

$$\lim_{k\to\infty} \frac{-3k^3 + 8k^2 - 7k + 11}{4k^3 - k^2 + 5} = \lim_{k\to\infty} \frac{a_k}{b_k} = -\frac{3}{4}.$$