Math 4606. Fall 2006.

Solutions to Exam 2

1. (25 points) Let A and B be two compact subsets of $\mathbb{R}^n$. Define the distance between A and B by

$$d(A, B) = \inf\{|x - y| : x \in A, y \in B\}.$$

Show that if $A \cap B = \emptyset$ then $d(A, B) > 0$.

Solution. Note that $d(A, B) \geq 0$. Suppose $d(A, B) = 0$. By the property of the infimum, there are sequences $\{x_k\}$ in A and $\{y_k\}$ in B such that

$$\lim_{k\to\infty} |x_k - y_k| = 0. \tag{1}$$

Since A and B are compact, there are subsequences $\{x_{k_j}\}$ and $\{y_{k_j}\}$ and $a \in A$ and $b \in B$, such that $\lim_{j\to\infty} x_{k_j} = a$ and $\lim_{j\to\infty} y_{k_j} = b$. By (1), $|a - b| = \lim_{j\to\infty} |x_{k_j} - y_{k_j}| = 0$. Hence $a = b \in A \cap B$ which contradicts the fact that $A \cap B = \emptyset$. Therefore we have $d(A, B) > 0$.

2. (25 points) Show that $f(x) = 2\sqrt{x} - 3\cos x + \ln(x^2 + 1)$ is uniformly continuous on $(1, \infty)$.

Solution. For $x \in (1, \infty)$, we have

$$f'(x) = \frac{1}{\sqrt{x}} + 3\sin x + \frac{2x}{x^2 + 1}.$$

Noting that $|2x| \leq x^2 + 1$ and $\sqrt{x} \geq 1$, we have

$$|f'(x)| \leq 1 + 3 + 1 = 5, \text{ for all } x \in (1, \infty).$$

Let $x, y \in (1, \infty)$, assume $x < y$, then by the mean value theorem, there is $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. Therefore

$$|f(y) - f(x)| \leq 5|y - x|, \qquad \text{for all } x, y \in (1, \infty).$$

Let $\varepsilon > 0$, take $\delta = \varepsilon/5$, for $x, y \in (1, \infty)$ and $|x - y| < \delta$, we have

$$|f(y) - f(x)| \leq 5|x - y| < 5\delta = \varepsilon.$$

Thus f is uniformly continuous on $(1, \infty)$.

3. (25 points) Let S be a connected set in $\mathbb{R}^3$ containing two points $(1,2,0)$ and $(-1,3,6)$. Show that S contains at least one point on the plane defined by the equation $3x - y + 2z - 5 = 0$.

Solution. Let $f(x,y,z) = 3x - y + 2z - 5$. Then f is continuous on S and $f(1,2,0) = 1 - 2 - 5 = -6 < 0$ and $f(-1,3,6) = -3 - 3 + 12 - 5 = 1 > 0$. Since S is connected, $(1,2,0)$ and $(-1,3,6)$ are in S, and the number 0 is between $f(1,2,0)$ and $f(-1,3,6)$, then by the intermediate value theorem, there is $(x_0, y_0, z_0) \in S$ such that $f(x_0, y_0, z_0) = 0$. Therefore (x_0, y_0, z_0) belongs to S and the plane given by $3x - y + 2z - 5 = 0$.

4. (25 points) Find the limit

$$\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4}.$$

Solution. Since $\lim_{x\to 0}(x^2 + 2\cos x - 2) = 0 = \lim_{x\to 0} x^4$, we can use L'Hôspital's rule (the last limit will verify the use of this rule). Applying the rule a few times, we obtain

$$\begin{aligned}
\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4} &= \lim_{x \to 0} \frac{2x - 2\sin x}{4x^3} \\
&= \lim_{x \to 0} \frac{2 - 2\cos x}{12x^2} \\
&= \lim_{x \to 0} \frac{2\sin x}{24x} \\
&= \lim_{x \to 0} \frac{2\cos x}{24} \\
&= \frac{1}{12}.
\end{aligned}$$