Math 4606. Spring 2007

Solutions to Sample Final Exam 1

Problem 1. Answer: some. If $A = [0,2]$, $B = [1,3]$, then $\sup A \cap B = 2 = \min(\sup A, \sup B)$.

On the other hand, if $A = \{0,1\}$ and $B = \{0,2\}$, then $\sup A \cap B = 0 \neq 1 = \min(\sup A, \sup B)$.

Problem 2. Without loss of generality, assume $|b| \geq |a|$. Then we need to show that $\sqrt{a^2+b^2} \leq \sqrt{2}|b|$. Indeed, $|b| \geq |a|$ is equivalent to $b^2 \geq a^2$, which implies $\sqrt{a^2+b^2} \leq \sqrt{2b^2} = \sqrt{2}|b|$.

Problem 3. This sequence is obtained by alternating the following three sequences

$$0.3, 0.3123, 0.3123123, \dots,$$
$$0.23, 0.23123, 0.23123123, \dots,$$
$$0.123, 0.123123, 0.123123123, \dots$$

Each of these sequences converges. The limits are the periodic infinite decimal fractions $0.3123123\dots$, $0.23123123\dots$, and $0.123123123\dots$, respectively.

Any convergent subsequence of the original sequence must have a infinite tail consisting of the terms of only one of the three sequences; otherwise, the subsequence will be divergent. Thus, the sublimits (which we also know as accumulation points) will be the three periodic infinite decimal fractions above.

Problem 4. Let $s_n = (-1)^n/n$ and $t_n = 1/n$. Both sequences converge to 0, while the quotient $s_n/t_n = (-1)^n$ diverges.

Problem 5. Given $\varepsilon > 0$, take $\delta = \varepsilon$. Then whenever $|x-1| < \delta$, we have $|f(x) - f(1)| = |f(x) - 1| = \begin{cases} 0, & \text{for } x \text{ rational,} \\ |x-1|, & \text{for } x \text{ irrational} \end{cases} \leq |x-1| < \delta = \varepsilon$. This means $f(x)$ is continuous at $x = 1$.

Problem 6. [After Mariam Kaynia] Take $\delta = 1/4$. Let us show it works. Suppose $|x-y| < 1/4$. If $x = y$, then $|\sqrt{x} - \sqrt{y}| = 0 < 1/2$, so that we can safely assume $x \neq y$. Then $|\sqrt{x} - \sqrt{y}| = |x-y|/(\sqrt{x}+\sqrt{y}) \leq |x-y|/|\sqrt{x}-\sqrt{y}|$, implying $|\sqrt{x}-\sqrt{y}|^2 \leq |x-y|$ by simple algebra. Taking the square root, we see $|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|} < \sqrt{1/4} = 1/2$.

Problem 7. False. Take $f(x) = -1$ and $g(x) = 1$. Then take your favorite function $h(x)$ enclosed between -1 and 1 which does not have a derivative at, say, $x = 0$. Or take $h(x) = \sin x$. Its derivative at $x = 0$ is 1, which is not equal to $f'(0) = 0$.

Problem 8. For $h > 0$

$$\frac{f(a+h) - f(a)}{h} \leq 0,$$

therefore, since the left-hand side has limit $f'(a)$ as $h \to 0+$, this limit must also be ≤ 0.

Problem X. For $f(x) = u(x) + iv(x)$, where u and v are the real and imaginary parts of f, respectively, the MVT would be equivalent to the following two equations satisfied simultaneously for some c between a and b:

$$\begin{aligned} u(b) - u(a) &= u'(c)(b-a), \\ v(b) - v(a) &= v'(c)(b-a). \end{aligned}$$

Take two real-valued functions $u(x)$ and $v(x)$, which have the c at two different points, for example, $u(x) = x^2$ and $v(x) = (x-1)(x+1)(x-1/2)$ on $[-1, 1]$. Then for $f(x) = u(x) + iv(x)$, we have $f(1) - f(-1) = 0$. If it happened that for some $c \in [-1, 1]$, we had $0 = f'(c)(1 - (-1))$, then $f'(c) = 0$, which would mean $u'(c) = 0$ and $v'(c) = 0$ simultaneously. However, $u'(c) = 2c = 0$ implies $c = 0$, and at $v'(0) \neq 0$ from a simple evaluation of the derivative.