Math 4606. Spring 2007

Solutions to Sample Final 2, Part II

Problem C. If $p(x)$ is a polynomial of even degree with the leading coefficient $a_{2n} > 0$,then $\lim_{|x|\to\infty} p(x) = \infty$. After that the argument is identical to the proof of Theorem 2.83a in the textbook.

Problem D. We will use sequential continuity, Theorem 1.15. If $p \neq 0$ or 1, then there will always be a sequence $\{a_n\}$ of rationals converging to p, for which the sequence $\{f(a_n)\}$ will converge to $1 - p$, and a sequence $\{b_n\}$ of irrationals converging to p, for which the sequence $\{f(b_n)\}$ will converge to $1-p^2$. Since for $p \neq 0$ or 1, we have $1 - p \neq 1 - p^2$, this implies that $f(x)$ will be discontinuous at such p.

If $p = 0$ or 1, then the values of $f(x)$ at both rationals and irrationals near p will be given by the two continuous functions, which both happen to converge to $f(p)$. Thus, if we have any sequence $\{c_n\}$ converging to p, then the sequence $\{f(c_n)\}$ will be made of two sequences, the rational subsequence and the irrational one, each of which converges to $f(p)$. Therefore, the sequence $\{f(c_n)\}$ will converge to $f(p)$.

Problem F. (i) Given $\varepsilon > 0$, take $\delta = \varepsilon/2M$. Then for each x, y, such that $|x - y| < \delta$, we have $|f(x) - f(y)| = |f'(c)(x - y)| \leq M\delta = \varepsilon/2 < \varepsilon$.

(ii) Take $\varepsilon = 1$. Then for any $\delta > 0$, take a natural number n such that $n(2n + 1) > 1/\delta$, so that $x = 1/n$ and $y = 1/(n + 1/2)$ will be less than δ apart. Then $|\sin \pi/x - \sin \pi/y| = |0 - \sin \pi(n + 1/2)| = 1 \geq \varepsilon$. This implies the function is not uniformly continuous.