PROBLEM SET 9

Due Wednesday, April 3, 2002 at the beginning of the class.

Study Sections 3.1 (pp. 90–91), 4.3, and 4.5 and solve the following problems.

Section 3.1: 6, 7
Section 4.2: 1
Section 4.3: 1 [Hint: This particular variant of Plateau's problem does not have a regular solution, because the disk with a pike qualifies as a solution among nonregular surfaces. One expects a regular solution to be close to the disk with a pike. Can you estimate the area of such surface?]
Section 4.5: 1

When two differentiable functions $f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy-Riemann equations

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial v}, \qquad \frac{\partial f}{\partial v} = -\frac{\partial g}{\partial u},$$

they are easily seen to be harmonic; in this situation f and g are said to be *harmonic conjugate*. Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parameterizations of minimal surfaces such that their component functions are pairwise harmonic conjugate; then $\mathbf{x}$ and $\mathbf{y}$ are called *conjugate minimal surfaces*. Prove that

(1) The helicoid and the catenoid are conjugate minimal surfaces.
(2) Given two conjugate minimal surfaces, $\mathbf{x}$ and $\mathbf{y}$, the surface

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y}$$

is again minimal for all $t \in \mathbb{R}$.
(3) All surfaces $\mathbf{z}$ above have the same first fundamental form, that is, E, F, and G.

Thus, any two conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces, and the first fundamental form of this family is independent of t.