SAMPLE MIDTERM EXAM - MATH 5378, SPRING 2002

THIS IS A CLOSED-BOOK, CLOSED-NOTES EXAM. YOU CAN USE IN YOUR SOLUTIONS ANY RESULT THAT WAS COVERED IN CLASS OR BY THE TEXT. YOU CAN ALSO USE ANY OF THE RESULTS FROM THE HOMEWORK.

(1) Let $\alpha : I \longrightarrow \mathbb{R}^2$ be a regular parameterized plane curve and $N(t)$ and $\kappa(t)$ be the normal vector and the curvature of α, respectively. Assume $\kappa(t) \neq 0$ for all $t \in I$. Recall that in this situation the curve

$$\mathcal{E}(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$$

is called the *evolute* of α. Show that the tangent line of the evolute is the normal line to α at t.

(2) Show that the knowledge of the vector function $B(s)$ (the binormal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the absolute value of the torsion $\tau(s)$ of α.

(3) One way to define a coordinate patch for the sphere S^2, given as $x^2 + y^2 + (z-1)^2 = 1$, is to consider the so-called *stereographic projection* $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ which carries a point $p = (x, y, z)$ on the sphere minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy-plane with the straight line which connects N to p. Let $(u, v) = \pi(x, y, z)$. (1) Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$\begin{aligned} x &= \frac{4u}{u^2 + v^2 + 4} \\ y &= \frac{4v}{u^2 + v^2 + 4} \\ z &= \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \end{aligned}$$

(2) Show it is possible to cover the sphere with two coordinate patches.

(4) Let $\lambda_1, \dots, \lambda_m$ be the normal curvature at $p \in M$ along unit directions making angles $0, 2\pi/m, \dots, (m-1)2\pi/m$ with a principal vector, $m > 2$. Prove that $\lambda_1 + \dots + \lambda_m = mH$, where H is the mean curvature at p. [Hint: Use the fact that for $\theta = 2\pi/m$

$$1 + \cos^2 \theta + \dots + \cos^2 (m-1)\theta = \frac{m}{2}.]$$

(5) Determine the umbilic points of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

[**Remark**: I want to remove this problem from the sample exam. This problem is a good problem for a homework, but bad for an exam, because the computations involved are too long. A straightforward way to do it

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would be to compute all those E, F, G, l, m, n, as it is done in Example 2.3 on p. 95 of the text, then compute K and H and solve the equation $H^2 = K$, which is equivalent for a point to be umbilic, see Exercise 1.5 on p. 90.

A second, more efficient, but still too long, way would be to notice that the vector $N_1 = (x/a^2, y/b^2, z/c^2)$ is normal, therefore it is equal to fN, for a unit normal N and $f = |N_1|$. Then observe that for any curve $\alpha(t) = (x(t), y(t), z(t))$ on the ellipsoid, a point is umbilic, if and only if it satisfies the equation

$$\left(\frac{dN_1}{dt} \times \frac{d\alpha}{dt}\right) \cdot N_1 = 0.$$

Multiply this equation by z/c^2 and express z and zz'/c^2 through x, y, x', y'. Then use the fact that the obtained equation should be satisfied for arbitrary x' and y', which gives a system of equations for x and y. There will be 12 solutions.]