SAMPLE SECOND MIDTERM EXAM PROBLEMS - MATH 5378, SPRING 2002

THIS WILL BE A CLOSED-BOOK, CLOSED-NOTES EXAM. YOU CAN USE IN YOUR SOLUTIONS ANY RESULT THAT WAS COVERED IN CLASS OR BY THE TEXT. YOU CAN ALSO USE ANY OF THE RESULTS FROM THE HOMEWORK.

Formulas for the first and second fundamental form and the Gaussian curvature from pp. 91 and 100, and the geodesic equations in the form

$$u'' + \Gamma^u_{uu}(u')^2 + 2\Gamma^u_{uv}u'v' + \Gamma^u_{vv}(v')^2 = 0,$$
$$v'' + \Gamma^v_{uu}(u')^2 + 2\Gamma^v_{uv}u'v' + \Gamma^v_{vv}(v')^2 = 0$$

will be provided on the exam.

(1) Compute the first fundamental form of the following surfaces of revolution:
   (a) $\vec{x}(u,v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$ (the torus);
   (b) $\vec{x}(u,v) = (av\cos u, av\sin u, bv)$ (the cone).

(2) Suppose we have surface parameterized by coordinates u and v whose first fundamental form is $E = 1$, $F = 0$, $G = u^2 + a^2$, where a is a constant. Find the area of the triangle bounded by the curves

$$u = \pm av, \quad v = 1.$$

(3) Show that the surface (the conoid)

$$\vec{x}(u,v) = (u\cos v, u\sin v, u + v)$$

is locally isometric to the surface (the hyperboloid of revolution)

$$\vec{y}(s,t) = (s\cos t, s\sin t, \sqrt{s^2 - 1}),$$

by a mapping given by

$$t = v + \arctan u, \quad s^2 = u^2 + 1.$$

(4) Compute the second fundamental form and the Gaussian curvature of the pseudosphere $\vec{x}(u,v) = (a\sin u\cos v, a\sin u\sin v, a(\ln\tan\frac{u}{2} + \cos u))$.

(5) Prove that a regular curve on a surface is a geodesic, if and only if its curvature is equal to the absolute value of its normal curvature.

(6) Two surfaces are tangent to each other along a common curve α. Prove that if α is a geodesic on one surface, then it will also be one on the other.

(7) Find the geodesics of a conic surface $\vec{x}(u,v) = u\alpha(v)$ with $|\alpha| = 1$ and $|\alpha'| = 1$.

(8) Find the geodesic curvature of a helix $u = b$ on the helicoid $\vec{x} = (u\cos v, u\sin v, av)$.

(9) Suppose on a sphere of radius R, a triangle formed by arcs of great circles is given. Let A be the area of the triangle. Find the sum of the interior angles of the triangle.

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(10) Show that on a minimal surface, the sum of the squares of the curvature and torsion of a geodesic are equal to $-K$. [Hint: Use the Frenet formula $N' = -\kappa T + \tau B$.]

(11) Show that on a simple (i.e., homeomorphic to an open disk) surface of $K \leq 0$, there are no closed geodesics.