SAMPLE FINAL PROBLEMS - MATH 5378, SPRING 2002

THIS WILL BE A CLOSED-BOOK, CLOSED-NOTES EXAM. YOU CAN USE IN YOUR SOLUTIONS ANY RESULT THAT WAS COVERED IN CLASS OR BY THE TEXT. YOU CAN ALSO USE ANY OF THE RESULTS FROM THE HOMEWORK.

(1) Assume $\alpha(s)$ is a unit speed curve of nonzero constant curvature whose normal lines are all perpendicular to a given constant vector. Show that α is part of either a circle or a standard circular helix, which is a curve $\beta(t)$ which, after a motion of the space, may be represented by an equation $\beta(t) = (a\cos t, a\sin t, bt)$, for $a \neq 0, b \neq 0$.

(2) Show that a space curve with $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for all s lies on a sphere if and only if

$$\left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2 = \text{const.}$$

(3) Compute the first fundamental form of the following surfaces:
  (a) $\vec{x}(u,v) = (\frac{a}{2}\left(u + \frac{1}{u}\right), \frac{b}{2}\left(u - \frac{1}{u}\right), v)$ (the hyperbolic cylinder);
  (b) $\vec{x}(s,t) = \alpha(s) + tB(s)$ (the binormal surface of a curve $\alpha(s)$ with a binormal $B(s)$).

(4) Show that a compact, regular, oriented surface M in $\mathbb{R}^3$ has a point with $K > 0$. Furthermore, show that if M is not homeomorphic to a sphere, then it also has a point with $K < 0$ and a point with $K = 0$.

(5) Show there is no surface in $\mathbb{R}^3$ with $E = G = \ell = 1$, $F = m = 0$ and $n = -1$.

(6) Let $\beta : I \longrightarrow \mathbb{R}^3$ be a curve of nonzero curvature and let $\vec{x}(u,v) = \beta(u) + v\beta'(u)$ be its tangent surface. Show that it is locally isometric to a cone (without the vertex).

(7) Compute the second fundamental form, the Gaussian and the mean curvatures of the following surface: $xyz = a^3$.

(8) Find the holonomy along a closed curve α on the unit sphere $x^2+y^2+z^2 = 1$, if
  (a) α is a parallel;
  (b) α consists of two meridians different by an angle ϕ and the part of the equator between them.

(9) Prove that on a surface of revolution, a meridian is always a geodesic.

(10) Let M be a surface in $\mathbb{R}^3$ such that all its geodesics are plane curves. Show that M is either a plane, or a sphere.

(11) Find the geodesics on a general cylindrical surface

$$\vec{x}(u,v) = (f(u), g(u), v),$$

where u is the arclength parameter of the base curve $(f(u), g(u))$.

(12) Describe the geodesics on $\mathbb{R}^2$, S^2, the cylinder $x^2 + y^2 = 1$.

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(13) Consider the catenoid $\vec{x}(u,v) = (u, \cosh u \cos v, \cosh u \sin v)$. Recall that in one of the homework problems you computed its Gaussian curvature as $K = -1/\cosh^4 u$. Show that there is exactly one simple closed geodesic on the catenoid.